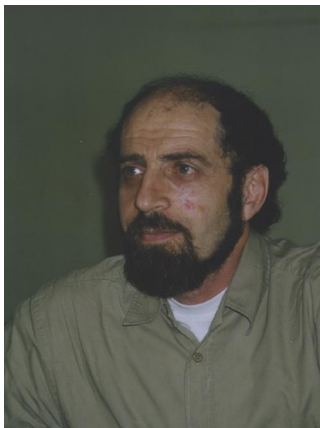


A history of the Zilber–Pink conjecture

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12th October 2026



Boris Zilber (1949–)

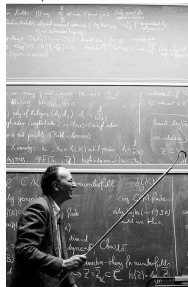


Richard Pink (1959–)

1922: starting point



Louis Joel Mordell (1888–1972)



Gerd Faltings (1954–)

- An elliptic curve is a smooth projective algebraic curve of genus 1.
- Mordell (1922) proved the Mordell–Weil theorem for elliptic curves: the group of rational points $E(\mathbb{Q})$ of an elliptic curve $E/\mathbb{Q}$ is a finitely generated abelian group.
- Recall that a height function on the points is the key ingredient of the proof.
- In the same paper Mordell conjectured that curves/ $\mathbb{Q}$ of genus > 1 have only finitely many rational points, now a theorem due to Faltings (1983), later reproved by Bombieri (1990) using a relatively elementary theory of heights after Vojta.

1938: Chabauty's conjecture

In his paper *Sur les équations diophantiennes liées aux unités d'un corps de nombre algébriques fini* (1938), Chabauty considered the following problem.

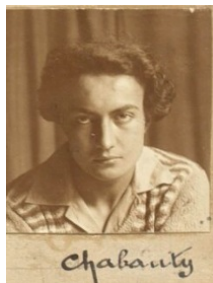
Let K be a number field of degree n and ω_i its integral basis. He wanted to solve for $x_i \in \mathbb{Z}$ the equation

$$\text{Norm}(x_1\omega_1 + \cdots + x_n\omega_n) = \pm 1, \quad (\dagger)$$

i.e. coefficients of the units of $\mathcal{O}_K$. The question is:

After imposing more simultaneous equations $F_i(x_1, \dots, x_n) = 0$, when are there infinitely many solutions?

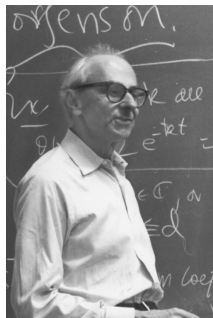
In other words, denoting $V = V(F_1, F_2, \dots, F_m) \subset \mathbb{A}^n$ and by Γ the solutions to $(\dagger)$, when is $V \cap \Gamma$ infinite? Motivated by Thue's theorem, Chabauty conjectured the answer to be **if and only if V , except maybe a finite number of points, is a union of sublattices of Γ and their cosets.**



"On peut conjecturer que ces conditions suffisantes sont aussi nécessaires..."

1960: Lang's formulation of Chabauty's conjecture

Let $G/\mathbb{C}$ be a torus (some $\mathbb{G}_m^n$) or an abelian variety, V a subvariety of G , and Γ a finitely generated subgroup of $G(\mathbb{C})$. If $V(\mathbb{C}) \cap \Gamma$ has infinitely many points, then V contains a finite number of translations of subgroup varieties of G , containing all but finitely many points of $V(\mathbb{C}) \cap \Gamma$.



Serge Lang (1927–2005)

Chabauty himself proved a related result (1941): if C is a curve of genus $g \geq 2$ over $\mathbb{Q}$ and $\text{rk } J(C)(\mathbb{Q}) < g$, then $C(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}$ is finite, hence $C(\mathbb{Q})$ is too.

On the other hand, the full Chabauty conjecture implies Mordell...

Chabauty implies Mordell

- 1 Take a curve C of genus $g \geq 2$, fix an embedding $i : C \rightarrow J(C)$, and take $G = J(C)$, $V = i(C)$, $\Gamma = J(C)(\mathbb{Q})$. Suppose per contra that $C(\mathbb{Q}) \subset i(C)(\mathbb{C}) \cap J(C)(\mathbb{Q})$ has infinitely many points.
- 2 Then by Chabauty, $i(C)$ contains a finite number of translations of subgroup varieties of $J(C)$, but $\dim i(C) = 1$ so $i(C)$ itself must be one;
- 3 however, a 1-dimensional subgroup variety of an abelian variety is an elliptic curve, which has genus $1 < g$, contradiction.

1965: Manin–Mumford

In the 60's, Manin was looking at Picard–Fuchs equations and the problem of the intersection of a curve with the torsion of its jacobian naturally arises. Mumford told this to Lang and the latter formulated a conjecture again.

Theorem (Manin–Mumford conjecture)

Let $G/\mathbb{C}$ be an abelian variety, V a closed irreducible subvariety of G , and G_{tor} the torsion points of $G(\mathbb{C})$. If $V \cap G_{\text{tor}}$ is Zariski dense in V , then V is a translation of an abelian subvariety by a point in G_{tor} .

Eventually proved by Raynaud in 1983.

In the same paper (1965), Lang reproduced a proof of Ihara–Serre–Tate of an elementary special case of Chabauty (if an irreducible plane curve V has infinitely many points whose coordinates are pairs of roots of unity, then V is defined by $x^r y^s = \zeta$ for $r, s \in \mathbb{Z}$ and ζ a root of unity).

Around the same time: Lang conjectures again

The Chabauty case Lang proved correspond to $G = \mathbb{G}_m^2$ and

$$\Gamma = \left\{ (z_1, z_2) \in \mathbb{G}_m^2(\mathbb{C}) = (\mathbb{C}^\times)^2 : \exists n, m \in \mathbb{Z}_{>0} : z_1^n = z_2^m = 1 \right\}.$$

In 1974, Liardet proved the same thing, replacing Γ by a more general

$$\Gamma = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \exists n, m \in \mathbb{Z}_{>0} : z_1^n, z_2^m \in \Gamma_0 \right\}$$

where Γ_0 is any finitely generated multiplicative subgroup of $\mathbb{C}$. Inspired by this, Lang proposed another conjecture.

Theorem (Mordell–Lang conjecture, 1991)

Let $G/\mathbb{C}$ be a semiabelian variety, V the same, Γ_0 a finitely generated subgroup of $G(\mathbb{C})$, and $\Gamma := \{g \in G(\mathbb{C}) : \exists n \in \mathbb{Z}_{>0} : ng \in \Gamma_0\}$ (the Γ above written additively), the “torsion hull” of Γ_0 . If $V \cap \Gamma$ is Zariski dense in V , then V is a translation of a semiabelian subvariety by a point in Γ .

Mordell–Lang conjecture

- Manin–Mumford is recovered by taking G to be an abelian variety and Γ_0 the trivial group.
- Liardet's theorem is recovered by taking G to be a torus and V a curve. Chabauty is recovered by considering the irreducible components of the Zariski closure of $V(\mathbb{C}) \cap \Gamma_0$.
- The statement for higher dimensions, that is, the full Mordell–Lang for tori, is a result due to Laurent (1984).
- If we again take Γ_0 to be the trivial group, but G any commutative group variety, the statement still holds true: this is a theorem of Hindry (1988).
- All of the above, together with work of Faltings and Vojta, led to the proof by McQuillan of the full Mordell–Lang in 1995.

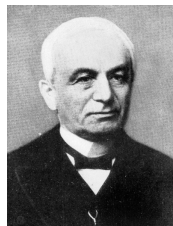
Things get modular

Consider the family of elliptic curves

$$E_j : y^2 = 4x^3 - \frac{27j}{j-1}x - \frac{27j}{j-1}$$

parameterised by $j \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. We want to know for which j does E_j have CM. Recall that this means $\text{End}(E_j)$ is an order $\mathcal{O}$ in an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

There are infinitely many such j . But if one bounds $[\mathbb{Q}(j) : \mathbb{Q}]$ the answer is finite. Indeed, $[\mathbb{Q}(j) : \mathbb{Q}]$ is the class number of $\mathcal{O}$ (Kronecker 1880s), and there are only finitely many discriminants with given class number (Heilbronn 1934).



Gauss (1798): “... series determinantum, quibus eadem classificatio data [...] respondet, semper abrumpi videtur [...] Demonstrationes autem *rigorosae* harum observationum perdifficiles esse videntur.”

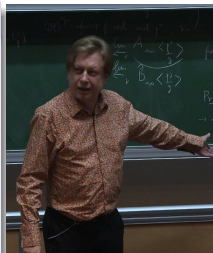
Kronecker (1882): “es war ein ganz neues, überraschendes und interessantes Phänomen, als mir bei der Beschäftigung mit der complexen Multiplication der elliptischen Functionen (im Winter 1856) Gattungen algebraischer Zahlen vor die Augen traten, welche in der angegebenen Weise mit den Gattungen von Quadratwurzeln negativer ganzer Zahlen associirt sind. Eine solche der Gattung $\sqrt{-n}$ associirte Gattung Γ liefert [...] die sämmtlichen, nach der *Kummerschen* Bezeichnung, idealen Divisoren der Gattung $\sqrt{-n}$; ihre Ordnung ist gleich der Classenanzahl für die Gattung $\sqrt{-n}$, und es haben überhaupt alle tieferen, auf die Composition und Classeneintheilung bezüglichen Eigenschaften der Gattung $\sqrt{-n}$ in den elementaren Eigenschaften der associirten Gattung Γ , so zu sagen, ihr Abbild.”

1989: André and his G-functions

Using G-functions, André was able to produce a higher dim analogue:

Theorem (1989)

Let $f : A \rightarrow C$ be a family of abelian varieties over a curve/ K , with some nice properties. In particular $A_{\bar{\eta}}$ is a simple abelian variety of odd dim $g > 1$. Then there are only finitely many s with $[K(s) : K] \leq d$ bounded such that there is no ring embedding $\text{End } A_s \hookrightarrow M_g(\mathbb{Q})$. In fact, the height $h(s)$ is bounded effectively by a power of $d + 1$.



Yves André (1959–)

- Bombieri's Hasse principle (1981): A set of algebraic numbers where there is a nontrivial relation between some G-functions has bounded height.
- Choosing basis ω_i for $H_{\text{dR}}^1(A/C)$ and γ_j for $R^1 f_* \mathbb{C}$, we get a matrix of periods $\Omega_{ij} = \int_{\gamma_j} \omega_i$. In coordinate of C , their Taylor expansions are G-functions.
- Construct nontrivial relations.

André–Oort conjecture

Shimura varieties are a natural inventory for base varieties that parameterise abelian varieties. For example,

- $Y(1)(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is a moduli space of elliptic curves, and
- $\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$ is a moduli space of principally polarised abelian varieties of dimension g .

The general notion for a point on a Shimura variety S whose fibre has CM is a **special point**. In Shimura theory, they are equivalent to **special subvarieties** of dim 0, and every special subvariety contains a dense subset of special points.

But the theorem above tells us that if a subvariety V has infinitely special points, the arithmetic of its points gets arbitrarily complex, and should have a geometric explanation. More precisely, the converse of the density of special points should be true.

Conjecture (André–Oort, 1st ver, 1989)

Let S be a Shimura variety and C a curve in S . If C contains infinitely many special points, then C is special.

What about Oort?

Coleman conjectured in 1987 that there are only finitely many complete smooth curves $C/\mathbb{C}$ of fixed genus $g \geq 4$ such that $J(C)$ has CM.

Oort found that the following conjecture can reduce Coleman to modular statement, now called the Coleman–Oort conjecture.

Theorem (André–Oort, 2nd ver, 1994)

If $V \subset \mathcal{A}_g$ contains a set of special points Zariski dense in V , then V is special.

So the natural generalisation of both is

Conjecture (André–Oort, final ver)

Let S be a Shimura variety and V an irreducible subvariety of S . If V contains a set of special points Zariski dense in V , then V is special.



Robert Frederick Coleman
(1954–2014)

Partial results of André–Oort



Bas Edixhoven (1962–2022)



David William Masser (1948–)

- Edixhoven did $S = Y(1)^2$ assuming GRH for imaginary quadratic fields (1996), then André unconditionally (1998) using Masser's work on transcendence (1975). (Edixhoven's method owes to Hindry.)
- His method was extended, eventually leading to a proof of the full André–Oort by Klingler–Ullmo–Yafaev (2014) under GRH for CM fields.
- Earlier, Pila introduced Pila–Zannier and pushed André's result to all $Y(1)^n$ (2011).
- Pila then did $\mathcal{A}_2$ with Tsimerman (2013) and the latter pushed to all $\mathcal{A}_g$ (2018).
- There is now apparently a full unconditional proof by Pila–Shankar–Tsimerman (announced 2021, latest draft 2024).

Turn of the millennium: unlikely intersections



Paula Tretkoff (1957–)

Unlikely intersections, where *unlikely* means dimensionally unexpected, were first investigated (1999) and named (2008) by Bombieri–Masser–Zannier.

The set

$\{\tau \in \overline{\mathbb{Q}} : \tau \text{ and } 1 - \tau \text{ are multiplicatively dependent}\}$ should be sparse when restricting degrees. Tretkoff (née Cohen) and Zannier (1998) showed an explicit height bound $\log 2$. BMZ's general theorem is



Umberto Zannier (1957–)

Theorem (1999a)

Let $C/\overline{\mathbb{Q}}$ be an irreducible curve in $\mathbb{G}_m^n$ for $n \geq 2$, not contained in any translate of a proper algebraic subgroup of $\mathbb{G}_m^n$. Then

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{H \subsetneq \mathbb{G}_m^n \text{ algebraic subgroup}} H(\overline{\mathbb{Q}})$$

has bounded height (effective).

But if one bounds the dimension...

BMZ noticed the following result (original context is reducibility of lacunary polynomials).

Theorem (Schinzel 1989)

Let $V \subset \mathbb{A}^n$ be a curve or a point. When $n \geq 3$, there is a constant $c(V)$ such that if

$\exists \zeta \in k^\times : (\zeta^{k_1}, \dots, \zeta^{k_n}) \in V$ for some $k_i \in \mathbb{Z}$, then either $\zeta^q = 1$ or $\exists M \in M_{2 \times n}(\mathbb{Z})$ with $\text{rk } M = 2$ and $v \in \mathbb{Z}^2$ such that $h(M) \leq c(V)$ and $(k_i) = vM$.



Andrzej Bobola Maria Schinzel
(1937–2021)

Inspired by this, BMZ wrote and proved a “striking” result.

Theorem (1999b)

Let C be a curve in $\mathbb{G}_m^n$ as in (a). Then

$$C(\overline{\mathbb{Q}}) \cap \bigcup_{\substack{H \subsetneq \mathbb{G}_m^n \text{ algebraic subgroup} \\ \dim H \leq n-2}} H(\mathbb{C})$$

is finite.

Around the same time: the conception of Zilber–Pink, and things get modular again

- Already in 1989, André was thinking about intersection of a subvariety of a Shimura variety with special subvarieties, but the absence of dimensional concerns led to a faulty conjecture.
- The year before, Pink finished his PhD at Bonn. His thesis develops the general theory of *mixed Shimura varieties*.
- In 2001 at the Franco–Taiwan Arithmetic Festival, in an attempt to unify André–Oort and Manin–Mumford, André first formulated the mixed André–Oort conjecture, which is however not enough to recover Mordell–Lang.
- In 2003, Pink proposed a tentative conjecture that tries to combine Mordell–Lang and André–Oort but ended up only implying a special case of latter (Hecke orbits, Edixhoven–Yafaev).
- Finally in 2005, having learned about BMZ’s work, Pink formulated Zilber–Pink for mixed Shimura varieties.

2005: Zilber–Pink conjecture for mixed Shimura varieties

Conjecture

Let S be a mixed Shimura variety and V an irreducible subvariety of S . If

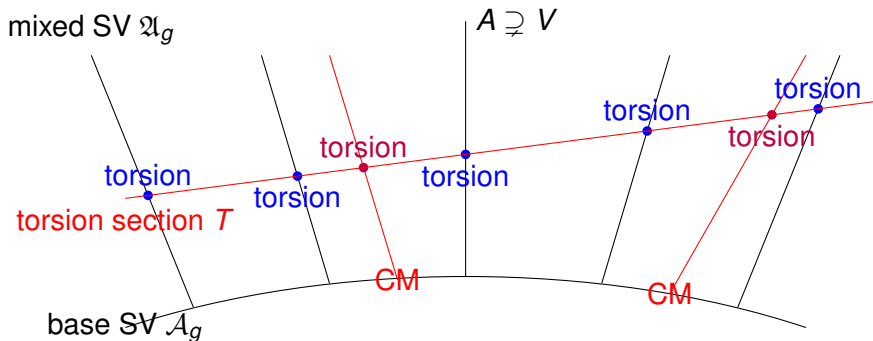
$$V \cap \bigcup_{\substack{Z \subset S \text{ special} \\ \dim Z + \dim V < \dim S}} Z$$

is Zariski dense in V , then V is contained in a proper special subvariety of S .

A mixed Shimura variety is roughly the universal family lying above a base Shimura variety; for example,

- the Legendre surface $L : y^2 = x(x-1)(x-\lambda)$ over $Y(2)$;
- the universal abelian $\mathcal{A}_g \rightarrow \mathcal{A}_g$.

Zilber–Pink implies Manin–Mumford



- $\dim \mathcal{A}_g = \frac{g(g+1)}{2} = \dim T$
- $\dim \mathfrak{A}_g = \dim \mathcal{A}_g + g = \frac{g(g+3)}{2}$
- $\dim V < \dim A = g$, so $\dim V + \dim T < \dim \mathfrak{A}_g$, an unlikely intersection

Zilber–Pink implies André–Oort

- 1 Let S be a MSV and $V \subsetneq S$ irreducible. Denote by $\mathcal{S}$ the set of special point in S .

- 2 In particular, $\dim V < \dim S$, so

$$V \cap \mathcal{S} = V \cap \bigcup_{\substack{Z \subset S \text{ special} \\ \dim Z = 0}} Z \subset V \cap \bigcup_{\substack{Z \subset S \text{ special} \\ \dim Z + \dim V < \dim S}} Z$$

is Zariski dense in V by assumption.

- 3 Hence by Zilber–Pink, write $V \subset T$ where $T \subsetneq S$ is special.
- 4 But then, considering T as a Shimura variety and denoting by $\mathcal{S}_T$ the special points of T , the intersection $V \cap \mathcal{S}_T = V \cap \mathcal{S}$ is again Zariski dense in V .
- 5 This iterative shrinking of the ambient Shimura variety must halt at when $V = T$ is special, since at each step the dimension drops strictly.

Partial results of Zilber–Pink

- The family $Y(1)^n$: $n = 1$ is trivial, $n = 2$ is André–Oort, $n = 3$ and curves C is the first interesting scenario.
 - ▶ Habegger–Pila (2012): $C/\overline{\mathbb{Q}}$ is *asymmetric*, that is, the restriction of coordinates $x_1, \dots, x_n$ to C have distinct degrees, up to at most one exception (ignoring constant ones) (Pila–Zannier)
 - ▶ Daw–Orr (2025): $\overline{C}^{\text{Zar}}$ in the compactification $(\mathbb{P}^1)^n$ of $Y(1)^n$ passes through the point $(\infty, \infty, \dots, \infty)$, or C is a line (G-functions)
 - ▶ Daw–Orr–Papas (2026): $n = 3$, $\overline{C}^{\text{Zar}} \setminus C$ contains a modular point
- The family $\mathcal{A}_g$: again, $n = 2$ and curves is the first case not covered by André–Oort.
 - ▶ Daw–Orr (2021, '22, '23): $\overline{C}^{\text{Zar}}$ intersects $\mathcal{A}_0$ in the compactification $\mathcal{A}_2 \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$ of $\mathcal{A}_2$
- Geometric Zilber–Pink: there is a finite set Σ of special subvarieties of S such that $\langle W \rangle_{\text{ws}}$ for any weakly optimal $W \subset V$ is a fibre of some $Y \in \Sigma$
 - ▶ Habegger–Pila (2016) for $Y(1)^n$
 - ▶ Daw–Ren (2018) for all Shimura varieties
 - ▶ Gao (2020) for mixed Shimura varieties of Kuga type

2016: Klingler's Hodge theoretic generalisation

In 2016 at the summer school on motives and complex multiplication in Ascona, Klingler proposed by far the most general version of the Zilber–Pink conjecture, for *variations of Hodge structures*.

The prototype is still a family of varieties $X \rightarrow S$. The Hodge filtrations of $H^n(X_s, \mathbb{C})$ vary, giving rise to a map $\phi : S^{\text{an}} \rightarrow \Gamma \backslash D$, which is holomorphic, horizontal, and locally liftable. In general we look at any S with such a map.

- A special subvariety of S looks like $\phi^{-1}(\Gamma' \backslash D')^\circ$.
- A special Z is *atypical* if
$$\dim \phi(Z) > \dim \phi(S) + \dim \Gamma' \backslash D' - \dim \Gamma \backslash D.$$

Conjecture (Zilber–Pink for variations of mixed Hodge structures)

The union of atypical special subvarieties (the *atypical locus*) of S is not Zariski dense in S , or equivalently, the atypical locus is a finite union.

Background

Conjecture (Hodge 1950)

Let $X/\mathbb{C}$ be smooth projective. Then every class in $H^{2c}(X, \mathbb{Q}) \cap H^{c,c}(X)$ is a rational combination of classes of X coming from subvarieties of codimension c .

Only known case is $c = 1$.

In the situation of a family $X \rightarrow S$, one can ask the relationship between

$$\{s \in S : X_s \text{ gains extra subvarieties}\}$$

and

$$\{s \in S : X_s \text{ has more Hodge classes than expected}\}.$$

Hodge says they are the same. Theory says the latter is the union of special subvarieties (the Hodge locus).



Weil (1977): “One may now ask whether imposing a certain Hodge class upon a generic member of an algebraic family of polarised algebraic varieties amounts to an algebraic condition upon the parameters. [...] It seems that a positive answer to that question would follow [...] from the Hodge conjecture (if true), but even from a weaker version of it [...]”



Eduardo Cattani (1946–), Pierre Deligne (1944–), Aroldo Kaplan (1942–2026)

- Cattani–Deligne–Kaplan (1995), landmark: the Hodge locus is a countable union of subvarieties.
- Brosnan–Pearlstein–Schnell (2010): pushed to admissible mixed.
- Zilber–Pink says the atypical part is moreover finite, hence algebraic. The typical part is either empty or dense in S , and there's a good criterion for which one happens (Baldi–Klingler–Ullmo 2023).