AN EXERCISE OF MILNE

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Abstract. This is Exercise 4-1 in [Mil22].

Theorem. Let G be a connected affine algebraic group over a field k of characteristic 0, and let H be an algebraic subgroup of G. Then H is normal in G if and only if, for every representation (V, r) of G and character χ of H, the eigenspace V_χ is stable under G.

Proof. $\Longleftarrow$ By Chevalley, there is a finite dimensional representation (V, r) of G and a 1 dimensional subspace L of V such that

$$H = \operatorname{stab}_G L = \{g \in G : \forall v \in L,\ r(g)v \in L\}.$$

Let $g \in G,\ h \in H$ and $v \in L$. Then by construction, $r(h)v \in L$. Our goal is $r(ghg^{-1})v \in L$. But r restricts to a representation of H and in particular a homomorphism $r_H : H \to \mathrm{GL}_L = \mathbb{G}_m$ again by construction, and we have $r = r_H$ on L. More explicitly, since L is 1 dimensional, $r(h)$ maps any vector in L to some nonzero scalar multiple of it, and we take that scalar to be $r_H(h)$. Note that r_H is also by definition a character of H, and by assumption, the eigenspace

$$V_{r_H} = \{v \in V : \forall h \in H,\ r(h)v = r_H(h)v\}$$

is G-stable, that is, if $r(h)v = r_H(h)v\ \forall h \in H$, then $r(h)r(g)v = r_H(h)r(g)v\ \forall h \in H$ as well for each $g \in G$. But note that this hypothesis is true when $v \in L$, that is, $L \subset V_{r_H}$, and we have

$$r(ghg^{-1})v = r(g)r(h)r(g^{-1})v = r(g)r_H(h)r(g^{-1})v = r(g)r(g^{-1})r_H(h)v = r_H(h)v \in L,$$

since $\mathbb{G}_m$ is the centre of GL_V.

$\Longrightarrow$ Let (V, r) be a representation of G and $\chi : H \to \mathbb{G}_m$ a character of H. Suppose $v \in V_\chi$, that is,

$$r(h)v = \chi(h)v\ \forall h \in H,$$

and let $g \in G$. We need to show that $r(g)v \in V_\chi$, that is,

$$r(h)r(g)v = \chi(h)r(g)v.$$

Since H is normal,

$$r(h)r(g)v = r(gg^{-1}hg)v = r(g)\chi(g^{-1}hg)v = \chi(g^{-1}hg)r(g)v,$$

where the last equality again follows from that χ takes values in the centre of GL_V. It remains to show $\chi(g^{-1}hg) = \chi(h)$. Consider the action of G on $\mathrm{Hom}(H, \mathbb{G}_m)$ given by

$$(g, \chi) \mapsto \left(\chi^g : h \mapsto \chi(g^{-1}hg)\right),$$

which gives us a morphism

$$\phi : G \to \mathrm{Aut}(\mathrm{Hom}(H, \mathbb{G}_m)) : g \mapsto (\chi \mapsto \chi^g)$$

and it suffices to see this is constant, since if $g \in H$ then clearly $\chi = \chi^g$. But since distinct characters are linearly independent (Corollary 4.24), $\mathrm{Aut}(\mathrm{Hom}(H, \mathbb{G}_m))$ has discrete topology, and since G is connected, the image of ϕ can only be a single point, as desired. □

References

[Mil22] J. S. Milne. *Algebraic groups. The theory of group schemes of finite type over a field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022. Zbl: 1390.14004.

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