MODULAR FORMS

JENS FUNKE, NOTES TYPED BY JIEWEI XIONG

Abstract. These are notes on modular forms based on Jens Funke's MAGIC (Mathematics Access Grid Instruction and Collaboration) lectures during Autumn 2025. The course comprises modular curves (as Riemann surfaces and as moduli space of elliptic curves over $\mathbb{C}$), modular functions and forms (basic properties, Eisenstein series, η-function), θ-series with arithmetic applications; modular forms and Dirichlet series with functional equation, Hecke operators and Petersson scalar product.

Contents

1. Tuesday 7 October 2025

Example 1.1 (A classical problem)**.** What is the number of ways a number N can be written as the sum of m squares, that is, what is

$$r_m(N) = \#\left\{(x_1, x_2, \dots, x_m) \in \mathbb{Z}^m : \sum_{i=1}^{m} x_i^2 = N\right\}?$$

We know

$$r_4(N) = 8 \sum_{\substack{d>0,\ d\mid N \\ 4\nmid d}} d \qquad = 8(p+1) \text{ if } N = p \text{ is prime}$$

and similar formulae for $r_2(N),\ r_6(N),\ r_8(N)$.

We could also ask for asymptotic formula for $N \to \infty$.

Example 1.2 (Three quadratic forms)**.**

$$\begin{aligned} P(x,y,u,v) &= x^2 + xy + 3y^2 + u^2 + uv + 3v^2 \\ &= \left(x + \frac{1}{2}y\right)^2 + \frac{11}{4}y^2 + \left(u + \frac{1}{4}y\right)^2 + \frac{11}{4}v^2 \end{aligned}$$

is evidently an integral and positive definite quadratic form. One also writes

$$2P(x,y,u,v) = \begin{pmatrix} x & y & u & v \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

where the matrix S has determinant 11^2.

Define the *representation number* of N by S as

$$r_s(N) := \#\left\{(x,y,u,v) \in \mathbb{Z}^4 : P(x,y,u,v) = N\right\},$$

a generalisation of 1.1. (This is indeed finite since it's the intersection of a compact set of $\mathbb{R}^4$ and a discrete set of $\mathbb{Z}^4$.)

Consider another quadratic form

$$Q(x,y,u,v) = 2(x^2+y^2+u^2+v^2)+2xu+xv+yu-2yv$$
$$= (x+u)^2+(y-v)^2+\left(x+\frac{1}{2}v\right)^2+\frac{3}{4}v^2+\left(y+\frac{1}{2}u\right)^2+\frac{3}{4}u^2$$

which is again integral and positive definite. The associated matrix is

$$T = \begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & -2 \\ 2 & 1 & 4 & 0 \\ 1 & -2 & 0 & 4 \end{pmatrix}$$

and $\det T = 11^2$ again. Similarly the associated matrix of the integral, positive definite

$$R(x,y,u,v) = x^2+4(y^2+u^2+v^2)+xu+4yu+3yv+7uv$$

is

$$U = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 8 & 4 & 3 \\ 1 & 4 & 8 & 7 \\ 0 & 3 & 7 & 8 \end{pmatrix}$$

with $\det U = 11^2$.

Two quadratic forms are *equivalent* if they differ by a change of basis for $\mathbb{Z}^4$. We have $r_A(N) = r_B(N)$ if $A \sim B$, that is, $CAC^t = B$ for some $C \in \mathrm{GL}_4(\mathbb{Z})$. It turns out that there are only three forms up to equivalence of determinant 11^2, represented by S, T, U above.

Note that the numbers are all even since if (x,y,u,v) represents N then $(-x,-y,-u,-v)$ does too.

Similarly we could ask:

(1) What's the exact formulae for $r_-(N)$ where $- = S, T$ or U?
(2) Asymptotic formulae for $r_-(N)$?
(3) Are there linear relationships between the $r_-(N)$'s?

It turns out that for (3),

$$\frac{3}{2}r_S(N) - \frac{1}{2}r_T(N) = r_U(N),$$

a result due to Hecke in the late 1930s. So we can forget about $r_U(N)$ and focus on the relationship between $r_S(N)$ and $r_T(N)$. Observe that $\frac{1}{4}(r_S(N) - r_T(N))$ grows slowly as N grows, and it's multiplicative (but $r_-(N)$'s themselves are not).

Example 1.3 (The first elliptic curve in nature)**.** Consider the elliptic curve $E : y^2+y = x^3-x^2$. Its Mordell–Weil group is $E(\mathbb{Q}) = \mathbb{Z}/5\mathbb{Z}$. It turns out that for all prime $p \neq 11$, we have $p+1-\#E(\mathbb{F}_p) = \frac{1}{4}(r_S(p) - r_T(p))$.

Definition 1.4 (For today)**.** A *modular form* is

(1) A holomorphic function f on the upper half plane $\mathbb{H} = \{z = x+iy : y > 0\}$.
(2) $f(z+1) = f(z)$.
(3) $f\left(-\frac{1}{Nz}\right) = \pm N^{\frac{k}{2}} z^k f(z)$, in which case we say f is of *level* N and *weight* k.
(4) The Fourier expansion of f starts at $n = 0$:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

(5) If $a_0 = 0$ we call f a *cusp form*.

Example 1.5 (Theta series)**.** For the quadratic form S in 1.2, set

$$\theta(z,S) = \sum_{x\in\mathbb{Z}^4} e^{2\pi i P(x) z} = \sum_{n\geq 0} r_s(n) e^{2\pi i n z},$$

which is a generating series for the $r_s(N)$'s. Same for T and U. By classical harmonic analysis,

$$\theta\left(-\frac{1}{11z}, S\right) = -11z^2\theta(z,S),$$

so it is indeed a modular form (of level 11 and weight 2).

Recall that r_S, r_T, r_U are linearly dependent, which means the space of modular forms of level 11 and weight 2 is 2-dimensional. We can also define an "easy" modular form of level 11 and weight 2:

$$E_2(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z} \quad \text{with} \quad b_p = p+1 \quad (p \neq 11 \text{ prime}),$$

the *Eisenstein series*, and hence write down some linear relations, for example

$$\frac{1}{4}\theta(z,S) + \frac{1}{6}\theta(z,T) = \frac{1}{4}\theta(z,T) + \frac{1}{6}\theta(z,U) = E_2(z),$$

which gives us

$$\frac{1}{4}r_S(n) + \frac{1}{6}r_T(n) = \frac{1}{4}r_t(n) + \frac{1}{6}r_U(n) = n+1 \quad \text{if } n \text{ is prime not 11.}$$

In this course we intend to study systematically theta series for any positive definite quadratic form as modular forms; for example,

(1) dimension of space of modular forms of fixed level and weight,
(2) construction of modular forms,
(3) Fourier coefficients as carrier of arithmetic information,
(4) multiplicativity of Fourier coefficients which leads to Hecke operators.

By "arithmetic information", again recall results from the elliptic curve picture:

Theorem 1.6 (Eichler–Shimura)**.** Given a modular cusp form of level N and weight 2 (and some more properties), one can construct an elliptic curve E of *conductor* N such that $p+1-\#E(\mathbb{F}_p) = a_p \quad (p \nmid N)$.

Theorem 1.7 (Taniyama–Shimura conjecture, proved by Wiles (1995) and more generally by Breuil—Conrad–Diamond–Taylor (2001))**.** The converse is true; that is, every elliptic curve is modular.

This implies the E in 1.3 is identified with the theta series $\frac{1}{4}(\theta(z,S) - \theta(z,T))$.

Example 1.8 (L-functions)**.** For $f = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ a modular (cusp) form of weight 2 (weight k is also okay), form its *L-series*

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad \Re s \gg 0.$$

This function has an analytic continuation to $\mathbb{C}$ and $L(f, 2-s) \leftrightarrow L(f,s)$ (similar to Riemann ζ-function). By 1.7, $L(f,s)$ gives us information about the elliptic curve E corresponding to the modular form f. In particular,

Conjecture 1.9 (Birch–Swinnerton-Dyer)**.** The order of the zero $s=1$ of $L(f,s)$ is the rank of the Mordell–Weil group $E(\mathbb{Q})$ where E is the associated elliptic curve of f.

The Hasse bound $|p+1-\#E(\mathbb{F}_p)| \leq 2\sqrt{p}$ is not a result of the theory of modular forms itself. But now with the connection given by 1.7, can we generalise this bound to modular forms of arbitrary k? It turns out the answer is yes and it's a consequence of Deligne's proof of the Weil conjectures. In particular with these bounds we can estimate the representation numbers we were interested in the first place (1.2).

2. Tuesday 14 October 2025

Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Denote the upper half plane

$$\mathbb{H} = \{\tau = u + iv : v > 0\}.$$

Lemma 2.1. *G acts on $\mathbb{H}$ via Môbius transformations, that is, by $g\tau = \frac{a\tau+b}{c\tau+d}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular we have $g_1(g_2\tau) = (g_1g_2)\tau$, $1\tau = \tau$ and $g\tau_1 = \tau_2 \iff \tau_1 = g^{-1}\tau_2$.*

Lemma 2.2. (1) *G acts on $\mathbb{H}$ transitively.*
(2) *$\mathrm{PSL}_2(\mathbb{R}) = \overline{G} = G/\{\pm 1\}$ acts effectively on $\mathbb{H}$, that is, only $\{\pm 1\} \in G$ acts trivially on $\mathbb{H}$.*

Proof. For (1) it's enough to consider $\tau_1 = i$, $\tau_2 = u + iv$, and

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \sqrt{v}^{-1} \end{pmatrix} i = \tau_2.$$

□

The action of G on $\mathbb{C}$ extends to $\mathbb{P}^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\}$.

Lemma 2.3. The differential is

$$\mathrm{d}(g\tau) = (c\tau + d)^{-2}\mathrm{d}\tau$$

Definition 2.4. Define $j(g,\tau) := c\tau + d$ as the *automorphy factor*, and we have $\Im(g\tau) = \frac{\Im\tau}{|j(g,\tau)|^2}$, $\mathrm{d}(g\tau) = \frac{1}{j(g,\tau)}\mathrm{d}\tau$.

Lemma 2.5. For $g, g' \in G$, we have the cocycle relation $j(gg',\tau) = j(g, g'\tau) \cdot j(g',\tau)$.

Definition 2.6. We call $\mathrm{SL}_2(\mathbb{Z})$ the *modular group.*

Proposition 2.7. The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Moreover, all the relations between the generators arise from $S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (ST)^3$. Note that the action corresponds to $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -\frac{1}{\tau}$. (The letter S for *Spiegelung*, German for reflection.)

Definition 2.8. For any subgroup Γ of G. define the equivalence relation on $\mathbb{H}$ via

$$\tau \sim \tau' \iff \exists \gamma \in \Gamma : \gamma\tau = \tau'.$$

A *fundamental domain* $\mathcal{F}$ for Γ is a closed connected region in $\mathbb{H}$ such that no two interior points are equivalent (identification at the boundary is allowed).

Theorem 2.9. A fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} : |\tau| \geq 1,\ -\frac{1}{2} \leq \Re\tau \leq \frac{1}{2} \right\}.$$

That is, every point in $\mathbb{H}$ is equivalent to a point in $\mathcal{F}$, and points in the interior of $\mathcal{F}$ are $\mathrm{SL}_2(\mathbb{Z})$-equivalent. The only identifications are: for $|\Re\tau| = \frac{1}{2}$ via $\tau \mapsto \tau \pm 1$; for $|\tau| = 1$ (the arc) via $\tau \mapsto -\frac{1}{\tau}$.

Proof. Apply T or T^{-1} repeatedly to move τ to the strip $-\frac{1}{2} \leq \Re\tau \leq \frac{1}{2}$. If now $|\tau| < 1$, apply S, then $\Im\left(-\frac{1}{\tau}\right) = \Im\left(\frac{\tau}{|\tau|^2}\right) > \Im\tau$, and we repeat the above two steps. This process must terminate, since there are only finitely many integer pairs c, d such that $|c\tau + d| < 1$.

Assume $\gamma\tau = \tau'$ and without loss of generality $\Im\tau' \geq \Im\tau$. Since $\Im(\gamma\tau) = \frac{\Im\tau}{|c\tau+d|^2}$ and $\Im\tau > \frac{1}{2}$, we must have $c \in \{0, \pm 1\}$. If $c = 0$, then T or T^{-1} gives identification at the boundary. If $c = 1$ ($-I$ acts trivially), then $d \in \{0, \pm 1\}$. □

Corollary 2.10. For $\tau \in \mathbb{P}^1(\mathbb{C})$, we let $\Gamma_\tau = \{\gamma \in \Gamma : \gamma\tau = \tau\}$ be its *isotopy subgroup* (stabiliser in Γ). Then all points τ in $\mathcal{F}$ have trivial isotopy group $\pm I$ except for the *elliptic (fix) points*: for $\tau = i$, $\pm\{I, S\}$, for $\tau = \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $\pm\{I, ST, (ST)^2\}$, and for $\tau = -\frac{1}{\omega}$, $\pm\{I, TS, (TS)^2\}$.

Proof of 2.7. Let $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$ be the group generated by S and T and let τ be an interior point of $\mathcal{F}$. Consider $\gamma\tau \in \mathbb{H}$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then by proof of the theorem there exists $\gamma' \in \Gamma'$ such that $\gamma'(\gamma\tau) \in \mathcal{F}$. Hence $\gamma'(\gamma\tau) = \tau$, so $\gamma'\gamma = \pm I$, thus $\gamma \in \Gamma'$ (up to sign). □

Definition 2.11. Let $\Gamma \in \mathrm{SL}_2(\mathbb{Z})$. For the set of Γ-equivalence classes, we write

$$X = X_\Gamma = \Gamma\backslash\mathbb{H}.$$

We have a quotient map

$$\pi : \mathbb{H} \to X$$

with endows X with the quotient topology, which is Hausdorff.

By unfolding the fundamental domain $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$, we see

Proposition 2.12. $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$ as topological spaces (and even true as noncompact complex Riemann curves/surfaces).

Definition 2.13. Let $\mathbb{P}^1(\mathbb{Q})$ be the set of *(rational) cusps.* For $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, call the set of Γ-equivalence classes of cusps the *cusps of* Γ. Write

$$\overline{\mathbb{H}} = \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}.$$

Lemma 2.14. $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set of cusps. In particular, ∞ is a representative of the single Γ-equivalence class of cusps for the modular group.

Proof. Write a rational number as $\frac{a}{c}$ with $(a, c) = 1$. Complete a and c to a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then $\gamma\infty = \frac{a}{c}$. □

We can extend the topology of $\mathbb{H}$ to $\overline{\mathbb{H}}$. We define a fundamental system of open neighbourhoods in $\overline{\mathbb{H}}$ for the cusps: for ∞, $U_C = \{\tau \in \mathbb{H} : \Im\tau > C\} \cup \{\infty\}$; for $\frac{a}{c} \in \mathbb{Q}$, γU_C with $\gamma\infty = \frac{a}{c}$. This translate of U_C gives a circle in $\mathbb{H}$ tangential to $\frac{a}{c}$.

Definition 2.15. The analytic structure at the cusp ∞ arises as follows. Define

$$\tau \mapsto q = e^{2\pi i\tau}$$

which maps $U_C \backslash \infty$ to the punctured disc of radius $e^{-2\pi C}$ around 0. Furthermore map

$$\infty \mapsto 0.$$

Then q descends to a map on $\pi_{\mathrm{SL}_2(\mathbb{Z})}(U_C)$ and is bijective for sufficiently large C ($C \geq 1$). So q is the local variable at ∞ for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 2.16. This makes $\overline{X} = \Gamma\backslash\overline{\mathbb{H}}$ into a compact Riemann surface.

3. Tuesday 21 October 2025

Definition 3.1. Let $k \in \mathbb{Z}$. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is *weakly modular of weight* k for the full modular group Γ_1 if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) = j(\gamma, \tau)^k f(\tau) \qquad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1.$$

Remark 3.2.
1. For generators T and S this means $f(\tau + 1) = f(\tau)$ and $f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$ respectively. In fact, by the cocycle relation, if f transforms as above for T and S, then it transforms accordingly for the whole group.
2. Consider $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $f(\tau) = (-1)^k f(\tau)$, so if k is odd then $f = 0$.

Definition 3.3. A *meromorphic modular form of weight* k is a weakly modular function that is meromorphic at ∞.

Remark 3.4 (What does meromorphicity at ∞ mean?)**.** From the Riemann surface point of view, while f is not a function on $X = \Gamma_1\backslash\mathbb{H}$, it is periodic and therefore defines a function in the local variable $q = e^{2\pi i\tau}$ in the punctured neighbourhood of ∞. We then require that f is meromorphic in q, that is,

$$f(\tau) = \sum_{n \gg -\infty} a_n q^n \qquad \Im\tau \gg 0,$$

(the q-expansion). We say f is *holomorphic* at ∞ if $a_n = 0$ $\forall n < 0$.

Definition 3.5. Let $k \in \mathbb{Z}$. A *(holomorphic) modular form of weight* k is a meromorphic modular form that is holomorphic on $\mathbb{H}$ and at ∞. Write $M_k(\Gamma_1)$ (resp. $S_k(\Gamma_1)$) for the vector space of modular forms (resp. cusp forms) of weight k for Γ_1. (The letter S for *Spitze*, German for cusp.)

Definition 3.6. A modular form is a *cusp form* if $a_0 = 0$.

Remark 3.7. A modular form f is not necessarily bounded as $v \to 0$, that is, on the real axis. This follows from considering

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \qquad \tau \to -\frac{d}{c} \in \mathbb{Q}$$

Lemma 3.8. Let $f \in M_k(\Gamma_1)$, then

$$v^{\frac{k}{2}} |f(\tau)|$$

is Γ_1-invariant.

Proof.

$$\Im(\gamma\tau)^{\frac{k}{2}} |f(\gamma\tau)| = \left(\frac{\Im\tau}{|c\tau + d|^2}\right)^{\frac{k}{2}} |c\tau + d|^k |f(\tau)| = \Im\tau^{\frac{k}{2}} |f(\tau)|.$$

□

Proposition 3.9. Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1)$. Then
1. $\Im\tau^{\frac{k}{2}} |f(\tau)|$ is bounded on $\mathbb{H}$.
2. $|a_n| \leq C n^{\frac{k}{2}}$ for some $C > 0$. (Hecke's bound)

Proof. For (1), note that $\Im\tau^{\frac{k}{2}}|f(\tau)|$ is bounded in the fundamental domain $\mathcal{F}$ since f is rapidly decreasing. Hence $\Im(\tau)^{\frac{k}{2}}|f(\tau)|$ is bounded on all of $\mathbb{H}$ by the previous lemma. So we have $|f(\tau)| \leq C'v^{-\frac{k}{2}}$ for some $C' > 0$. For (2), we then see

$$|a_n| = \left|\int_{-\frac{1}{2}}^{\frac{1}{2}} f(u+iv)e^{-2\pi in\tau}\mathrm{d}u\right| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\tau)|e^{2\pi nv}\mathrm{d}u \leq C'v^{-\frac{k}{2}}e^{2\pi nv} \qquad \forall v.$$

Now take $v = \frac{1}{n}$, then $|a_n| \leq C'e^{2\pi}n^{\frac{k}{2}}$. □

Remark 3.10. The Ramanujan–Petersson conjecture, proved by Deligne as a consequence of the Weil conjectures, states

$$|a_n| = O\left(n^{\frac{k-1}{2}+\varepsilon}\right),$$

that is, for all $\varepsilon > 0$, there is a $C = C(\varepsilon)$ such that $|a_n| \leq Cn^{\frac{k-1}{2}+\varepsilon}$.

Definition 3.11. Let $k \geq 4$ be an even integer. Define the *Eisenstein series* of weight k for $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ by

$$G_k(\tau) := \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k}.$$

Here $\sum'$ means we only sum over pairs $(m,n) \neq (0,0)$.

Lemma 3.12. The Eisenstein series $G_k(\tau)$ converges absolutely and uniformly in compact subsets of $\mathbb{H}$ (in particular is holomorphic on $\mathbb{H}$). Moreover,

$$G_k(\tau) \in M_k(\Gamma_1),$$

and

$$G_k(\infty) = 2\zeta(k)$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann zeta function.

Proof. For the first statement, fix τ and a lattice $L_\tau = \{m\tau + n : m, n \in \mathbb{Z}\}$ in $\mathbb{C}$. For $N > 0$, let M_N be the boundary of the parallelogram with vertices $\pm N\tau + N$. We have $|M_N \cap L| = 8N$ ($2N + 1$ points on each side with double counts on the 4 vertices). Let $r(\tau)$ be the distance from 0 to M_1. Then the distance from 0 to M_N is $Nr(\tau)$. As τ varies in a compact set we have $R \leq r(\tau)$ for some $R > 0$. Thus

$$|G_k(\tau)| \leq \sum_{\omega\in L_\tau\setminus\{0\}} |\omega|^{-k} = \sum_{N=1}^{\infty}\sum_{\omega\in L_\tau\cap M_N} |\omega|^{-k} \leq 8\sum_{N=1}^{\infty} N(Nr(\tau))^{-k} \leq 8\zeta(k-1)R^{-k},$$

so by the Weierstrass M-test we are done.

Now by definition $G_k(\tau+1) = G_k(\tau)$. More subtly, by absolute convergence

$$G_k\left(-\frac{1}{\tau}\right) = \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{-\frac{m}{\tau}+n}^k = \tau^k \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{(-m+\tau n)^k} = \tau^k G_k(\tau).$$

Hence

$$\lim_{\tau\to\infty} \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k} = \lim_{\tau\to\infty} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^k} = 2\zeta(k)$$

as desired. □

Theorem 3.13. For $k \geq 4$ even,

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where

$$\sigma_{k-1}(n) = \sum_{\substack{m|n,\\ m>0}} m^{k-1}$$

is the power divisor function.

Proof. The key is the Lipschitz summation formula

$$(\dagger)\qquad \sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{N=1}^{\infty} N^{k-1}q^N$$

for any $k \geq 2$. We prove this later. Assuming this, we have

$$\sum_{n,m\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k} = \sum_{n\neq 0}\frac{1}{n^k} + 2\sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^k}\right) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{N=1}^{\infty} N^{k-1}q^{mN}.$$

□

Theorem 3.14 (Poisson summation formula)**.** Let f be "nice", for example $C^2(\mathbb{R})$, and $f, f', f'' = O(x^{-1+c})$ for some $c > 0$ (so that its periodisation $F(x) \coloneqq \sum_{n\in\mathbb{Z}} f(x+n)$ converges absolutely and is differentiable in x). Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$$

where

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x y}\mathrm{d}x$$

is the Fourier transform.

Proof sketch. $F(x)$ is periodic, thus under the niceness condition can be written as a Fourier series

$$\begin{aligned}\sum_{N=-\infty}^{\infty}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} F(y)e^{-2\pi i N y}\mathrm{d}y\right)e^{2\pi i N x} &= \sum_{N=-\infty}^{\infty}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{n\in\mathbb{Z}} f(y+n)e^{-2\pi i N y}\mathrm{d}y\right)e^{2\pi i N x}\\ &= \sum_{N=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)e^{-2\pi i N y}\mathrm{d}y\right)e^{2\pi i N x}\\ &= \sum_{N=-\infty}^{\infty}\widehat{f}(N)e^{2\pi i N x},\end{aligned}$$

and specialise $x = 0$. □

Proof of †*.* We apply Poisson summation to the function $f_\tau(x) = (\tau + x)^{-k}$. Then for the Fourier transform, we see

$$\widehat{f_\tau}(N) = \int_{-\infty}^{\infty}(\tau+x)^{-k}e^{-2\pi i x N}\mathrm{d}x = \int_{z=-\infty+iv}^{\infty+iv} z^{-k}e^{-2\pi i N z}\mathrm{d}z.$$

Consider the integral over the (large) rectangle $R = R_v(u, y)$ with vertices $\pm u + iv$ and $\pm u - iy$ for some $u, y > 0$. The integrand $z^{-k}e^{-2\pi i N z}$ is meromorphic with a singularity at $z = 0$ and

$$-2\pi i \operatorname{Res}_{z=0}\left(z^{-k}e^{-2\pi i N z}\right) = \frac{(-2\pi i)^k}{(k-1)!}N^{k-1}.$$

By the residue theorem, we have

$$\int_{R_v(u,y)} z^{-k}e^{-2\pi i N z}\mathrm{d}z = 2\pi i \operatorname{Res}_{z=0}\left(z^{-k}e^{-2\pi i N z}\right).$$

But now for $u, y \to \infty$, the integral at the upper side becomes $-a_N$, while the integrals over the other sides vanish, since the integrand vanishes uniformly in the limit. For $N = 0$, the integral values itself to 0. For $N < 0$, an argument similar as above works: use a rectangle not including the origin (exercise).

There exist shorter proofs, but Poisson summation also works in more general situations. □

Remark 3.15. We have

$$\zeta(k) = -(2\pi i)^k\frac{B_k}{2(k!)},$$

where B_k is the kth Bernoulli number, which are rational and defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_k\frac{x^k}{k!}.$$

The first few values are

$$B_2 = \frac{1}{6},\quad B_4 = -\frac{1}{30},\quad B_6 = \frac{1}{42},\quad B_8 = -\frac{1}{30},\quad B_{10} = \frac{5}{66},\quad B_{12} = -\frac{691}{2730},\quad B_{14} = \frac{7}{6}.$$

Definition 3.16. The *normalised Eisenstein series* $E_k(\tau)$ is defined by

$$E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

In particular, $E_k(\tau)$ has rational Fourier coefficients.

Lemma 3.17.

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{n,m\in\mathbb{Z},\\ (n,m)=1}} \frac{1}{(m\tau+n)^k}.$$

Proof. We have

$$\{(n,m) \neq (0,0)\} = \coprod_{c=1}^{\infty} \{(cn, cm) : (n,m) = 1\}.$$

Hence

$$G_k(\tau) = \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k} = \sum_{c=1}^{\infty} \sum_{\substack{n,m\in\mathbb{Z},\\ (n,m)=1}} \frac{1}{(cm\tau+cn)^k} = \zeta(k) \sum_{\substack{n,m\in\mathbb{Z},\\ (n,m)=1}} \frac{1}{(m\tau+n)^k}.$$

□

Lemma 3.18. The correspondence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c,d)$ gives a bijection between $\Gamma_\infty \backslash \Gamma_1$ and the set

$$\{(n,m) \in \mathbb{Z} : (n,m) = 1\}/\{\pm 1\}.$$

4. Tuesday 28 October 2025

Definition 4.1. Let k be an even integer. Define the *Eisenstein series* of weight k and a complex parameter s for $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$

$$G_k(\tau, s) := \sum_{n,m\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k |m\tau+n|^{2s}}.$$

Also define the *normalised Eisenstein series*

$$E_k(\tau, s) := \frac{1}{2} \sum_{\substack{n,m\in\mathbb{Z},\\ (n,m)=1}} \frac{1}{(m\tau+n)^k |m\tau+n|^{2s}}.$$

Lemma 4.2. The Eisenstein series $G_k(\tau,s)$ and $E_k(\tau,s)$ converge absolutely for $\Re s > 1 - \frac{k}{2}$ and define holomorphic functions in s in this range. Moreover,

$$G_k(\gamma\tau, s) = j(\gamma,\tau)^k |j(\gamma,\tau)|^{2s} G_k(\tau, s) \qquad \forall \gamma \in \Gamma_1$$

and same for $E_k(\tau,s)$. Finally,

$$G_k(\tau,s) = 2\zeta(k+2s) E_k(\tau,s).$$

Remark 4.3. The Eisenstein series $E_k(\tau,s)$ are of fundamental importance for the general theory. However, they are often not covered in the beginning textbooks. One property is that they have a meromorphic continuation in s to the entire complex plane. Of particular interest are the cases weight $k = 0$ and 2.

Proposition 4.4. The limits $\lim_{s\to 0^+} G_2(\tau,s)$ and $\lim_{s\to 0^+} E_2(\tau,s)$ exist, and define

$$G_2^*(\tau) := \lim_{s\to 0^+} G_2(\tau,s), \qquad E_2^*(\tau) := \lim_{s\to 0^+} E_2(\tau,s).$$

Then $G_2^*(\tau)$ and $E_2^*(\tau)$ are (mildly) nonholomorphic modular forms of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ and their Fourier expansions are given by (writing $\tau = u + iv$)

$$G_2^*(\tau) = -\frac{\pi}{v} + \frac{1}{3}\pi^2 - 8\pi^2 \sum_{n=1} \sigma_1(n) q^n,$$

$$E_2^*(\tau) = -\frac{3}{\pi v} + 1 - 24 \sum_{n=1} \sigma_1(n) q^n.$$

The introduction of $E_2^*(\tau)$ in this way is the *Hecke summation* or *Hecke's trick*.

Corollary 4.5. We can define $G_2(\tau)$ and $E_2(\tau)$ by

$$G_2(\tau) = \frac{1}{3}\pi^2 - 8\pi^2 \sum_{n=1} \sigma_1(n)q^n, \qquad E_2(\tau) = 1 - 24\sum_{n=1} \sigma_1(n)q^n.$$

Note that this is exactly the formula of the Fourier expansions of G_k and E_k from the previous lecture, when (formally) specialised to $k = 2$. Then (by 4.4),

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d),$$
$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6}{\pi} i c(c\tau + d).$$

Remark 4.6. The definition of G_2 arises from fixing an order of summation and the subsequent calculation of the Fourier expansion as in the higher weight case:

$$G_2(\tau) = \sum_{m\in\mathbb{Z}} \sideset{}{'}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau) + n^2}.$$

This double series is conditionally convergent. Applying $S\tau = -\frac{1}{\tau}$ gives formally

$$\tau^2 \sum_{m\in\mathbb{Z}} \sideset{}{'}\sum_{n\in\mathbb{Z}} \frac{1}{(n\tau + m)^2}.$$

So 4.5 gives the discrepancy which occurs when the two summations are switched.

Definition 4.7. For $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$ as usual, define the *Dedekind η-function*

$$\eta(\tau) := q_{24} \prod_{n=1}^{\infty} (1 - q^n) \qquad q_{24} = q^{\frac{1}{24}} = e^{\frac{\pi i}{12}}$$

Define the *discriminant function*

$$\Delta(\tau) := \eta(\tau)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Remark 4.8. Since $\sum_{n=1}^{\infty} \log(1 - q^n)$ converges absolutely and uniformly in compact subsets, the infinite product $\prod_{n=1}^{\infty}(1 - q^n)$ defines a holomorphic function on $\mathbb{H}$. Moreover, $\eta(\tau) \neq 0$ and $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$.

Proposition 4.9. We have

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

where $\sqrt{-}$ denotes the principal branch of the square root (which makes sense as $-i\tau$ lies in the right half plane).

Proof. Consider the logarithmic derivative

$$\frac{\partial}{\partial\tau}\log(\eta(\tau)) = \frac{\pi i}{12} - 2\pi i \sum_{d=1}^{\infty} \frac{dq^d}{1 - q^d} = \frac{\pi i}{12} - 2\pi i \sum_{d=1}^{\infty} d \sum_{m=1}^{\infty} q^{md}$$
$$= \frac{\pi i}{12} - 2\pi i \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d\right) q^n = \frac{\pi i}{12} E_2(\tau).$$

Thus

$$\frac{\partial}{\partial\tau}\log\left(\eta\left(-\frac{1}{\tau}\right)\right) = \frac{\pi i}{12}\tau^{-2} E_2\left(-\frac{1}{\tau}\right),$$

and

$$\frac{\partial}{\partial\tau}\log\left(\sqrt{-i\tau}\,\eta(\tau)\right) = \frac{1}{2\tau} + \frac{\pi i}{12}E_2(\tau) = \frac{\pi i}{12}\left(E_2(\tau) + \frac{12}{2\pi i\tau}\right).$$

But by 4.5, the right hand sides of the last two equations coincide. Hence

$$\frac{\partial}{\partial\tau}\log\left(\eta\left(-\frac{1}{\tau}\right)\right) = \frac{\partial}{\partial\tau}\log\left(\sqrt{-i\tau}\,\eta(\tau)\right),$$

thus
$$\eta\left(-\frac{1}{\tau}\right) = C\sqrt{-i\tau}\eta(\tau) \qquad \text{for a constant } C \text{ independent of } \tau,$$
but $C = 1$ for $\tau = i$. □

Corollary 4.10. $\Delta(\tau) = S_{12}(\Gamma_1)$ with $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$.

Proof. For T, $\Delta(\tau+1) = \Delta(\tau)$ is clear, and for S,
$$\Delta\left(-\frac{1}{\tau}\right) = \eta^{24}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}^{24}\eta(\tau)^{24} = \tau^{12}\Delta(\tau).$$
Clearly $\Delta(\infty) = 0$. □

Definition 4.11. Denote the Fourier expansion of Δ by
$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$
(The first few values of $\tau(n)$ are 1, −24, 252, −1472, 4830, −6048, −16744, 84480, −113643, −115920.) Ramanujan conjectured

(1) $\tau(mn) = \tau(m)\tau(n)$ for $(n,m) = 1$, proved by Mordell (1919), generalised by Hecke to Hecke operators.
(2) $|\tau(p)| \leq 2p^{\frac{11}{2}}$, generalised to all cusps forms by Petersson, proved by Deligne (1969–1973).
(3) Some congruences, e.g. $\tau(n) = \sigma_{11}(n) \bmod 691$.
(4) Lehmer's conjecture: $\tau(n) \neq 0\ \forall n$ (still open).

Proof of 4.4. we have
$$G_2(\tau,s) = \sum_{n\neq 0}\frac{1}{n^2|n|^{2s}} + 2\sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2|m\tau+n|^{2s}}\right) \qquad \Re s > 0.$$
The 0th Fourier coefficient of the inner summand is given by
$$I_s(m\tau) = \int_0^1\sum_{n\in\mathbb{Z}}\frac{\mathrm{d}t}{(m\tau+n+t)^2|m\tau+n+t|^{2s}} = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(m\tau+t)^2|m\tau+t|^{2s}} \qquad \Re s > -\frac{1}{2}.$$
Now
$$G_2(\tau,s) - 2\sum_{m=1}^{\infty}I_s(m\tau) = \sum_{n\neq 0}\frac{1}{n^2|n|^{2s}} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\left(\frac{1}{(m\tau+n)^2|m\tau+n|^{2s}} - \int_n^{n+1}\frac{\mathrm{d}t}{(m\tau+t)^2|m\tau+t|^{2s}}\right)$$
where the right hand side converges absolutely and uniformly in compact subsets for $\Re s > -\frac{1}{2}$.

Indeed for $f(t) = (m\tau+t)^{-2}|m\tau+t|^{-2s}$ we have the complex mean value theorem
$$\left|f(n) - \int_n^{n+1}f(t)\ \mathrm{d}t\right| \leq \int_n^{n+1}|f(n)-f(t)|\ \mathrm{d}t \leq \int_n^{n+1}\max_{u\in[n,n+1]}|f'(u)|\ \mathrm{d}t = \max_{u\in[n,n+1]}|f'(u)|.$$
Hence the summand of the double sum is $O\left(|m\tau+n|^{-3-2s}\right)$, so the limit of the right hand side as $s\to 0$ exists and can be computed by evaluation at $s = 0$. Using
$$\int_n^{n+1}\frac{\mathrm{d}t}{(m\tau+t)^2} = -(m\tau+n+1)^{-1} + (m\tau+n)^{-1},$$
we easily obtain $G_2(\tau)$ for the right hand side at $s = 0$.

It remains to compute
$$\lim_{s\to 0}2\sum_{m=1}^{\infty}I_s(m\tau).$$
For $\Re s > -\frac{1}{2}$ and writing $\tau = u + iv$,
$$I_s(\tau) = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(u+t+iv)^2|u+t+iv|^{2s}} = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(t+iv)^2(t^2+v^2)^s} = \frac{I(s)}{v^{1+2s}}$$
with $I(s) = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(t+i)^2(t^2+1)^s}$. Thus
$$2\sum_{m=1}^{\infty}I_s(m\tau) = \frac{2}{v^{1+2s}}\zeta(1+2s)I(s).$$

Now the Laurent expansion for $\zeta(1+2s)$ at $s=0$ is given by

$$\zeta(1+2s) = \frac{1}{2s} + O(1).$$

For $I(s)$, we have $I(0)=0$, while

$$I'(0) = -\int_{-\infty}^{\infty} \frac{\log(t^2+1)}{(t+i)^2}\,\mathrm{d}t = \left(\frac{1+\log(t^2+1)}{t+i} - \arctan(t)\right)\Bigg|_{-\infty}^{\infty} = -\pi.$$

So $I(s) = -\pi s + O(s^2)$. Thus

$$\lim_{s\to 0} \frac{2}{v^{1+2s}} \zeta(1+2s) I(s) = -\frac{\pi}{v}.$$

□

5. Tuesday 4 November 2025

Lemma 5.1. Let $k \geq 4$ be even. Then

$$M_k(\Gamma_1) = S_k(\Gamma_1) \oplus \mathbb{C}E_k.$$

Proof. For $f \in M_k(\Gamma_1)$ with constant coefficient a_0, we write

$$f = (f - a_0 E_k) + a_0 E_k,$$

then $f - a_0 E_k$ is a cusp form. Since $S_k(\Gamma_1) \cap \mathbb{C}E_k = 0$, the result follows. □

Lemma 5.2. The assignment $f \mapsto \frac{f}{\Delta}$ induces an isomorphism

$$S_k(\Gamma_1) \simeq M_{k-12}(\Gamma_1), \qquad S_k(\Gamma_1) = \Delta M_{k-12}(\Gamma_1).$$

Proof. Let $f \in S_k(\Gamma_1)$. Since Δ has no zeros on $\mathbb{H}$, $\frac{f}{\Delta}$ is holomorphic on $\mathbb{H}$. Since f is a cusp form, i.e. vanishes at ∞, the quotient $\frac{f}{\Delta}$ is also holomorphic at ∞ (as Δ has a simple 0 at ∞). It is clear that $\frac{f}{\Delta}$ transforms of weight $k-12$. Thus $\frac{f}{\Delta} \in M_{k-12}(\Gamma_1)$. The inverse map is given by $f \mapsto \Delta f$, which maps $M_{k-12}(\Gamma_1)$ to $S_k(\Gamma_1)$. □

Definition 5.3. For f a meromorphic modular form of weight k and $P \in \overline{\mathbb{H}} = \mathbb{H} \cup \infty \cup \mathbb{Q}$, we denote by $\mathrm{ord}_P(f)$ the order of vanishing of f at P (minus the order of the pole of f at P). Note that $\mathrm{ord}_P(f)$ does not change if P is replaced by γP for $\gamma \in \Gamma_1$. So $\mathrm{ord}_\infty(f)$ is the index of the first nonvanishing Fourier coefficient of f.

Let N_P for $P \in \overline{\mathbb{H}}$ be the order of stabiliser of P inside $\overline{\Gamma_1}$. So in the fundamental domain $\mathcal{F}$, we have

$$N_P = \begin{cases} 2 & \text{if } P = i, \\ 3 & \text{if } P = \omega = e^{\frac{2\pi i}{3}}, \\ 1 & \text{else.} \end{cases}$$

Proposition 5.4 ($\frac{k}{12}$-formula)**.** Let f be a nonzero meromorphic modular form of weight k for Γ_1. Then

$$\sum_{P \in \Gamma_1 \backslash \overline{\mathbb{H}}} \frac{1}{N_P} \mathrm{ord}_P(f) = \frac{k}{12}$$

Explicitly,

$$\mathrm{ord}_\infty(f) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\omega(f) + \sum_{\substack{P \in \Gamma_1 \backslash \mathbb{H}, \\ P \neq i, \omega}} \mathrm{ord}_P(f) = \frac{k}{12}.$$

Corollary 5.5. $\dim M_k(\Gamma_1) = 0$ for $k < 0$, $M_0(\Gamma_1) = \mathbb{C}$, and

$$\dim M_k(\Gamma_1) \leq \begin{cases} \left\lfloor \frac{1}{2} \right\rfloor + 1 & \text{if } k \neq 2 \bmod 12, \\ \left\lfloor \frac{1}{2} \right\rfloor & \text{if } k = 2 \bmod 12. \end{cases}$$

Proof. For f holomorphic all terms on the left hand side of 5.4 are nonnegative. This gives vanishing for $k < 0$. For $k = 0$, let c be a value taken by $f \in M_0(\Gamma_1)$. Then the left hand side of 5.4 for $g(\tau) = f(\tau) - c$ is positive, but the right hand side vanishes.

Let $m = \lfloor \frac{1}{2} \rfloor + 1$ and choose m distinct nonelliptic fixed points $P_i \in \mathbb{H}$. Given modular forms $f_1, \ldots, f_{m+1} \in M_k(\Gamma_1)$, we can find a linear combination f which vanishes at all P_i. So the left hand side of 5.4 gives at least $m = \lfloor \frac{1}{2} \rfloor + 1$, and f must be 0. Thus any $m + 1$ vectors are linearly dependent, and $\dim M_k(\Gamma_1) \leq m$.

For $k = 2 \bmod 12$, we can improve the estimate by 1: note that the only way to satisfy 5.4 is to have a simple zero at i and a double zero at ω (contributing a total of $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ to the left hand side of 5.4). Hence we need $m - 1$ further zeros, and argue as before. □

Theorem 5.6. Let $k > 0$ be even, then

$$\dim M_k(\Gamma_1) = \begin{cases} \left\lfloor \dfrac{1}{2} \right\rfloor + 1 & \text{if } k \neq 2 \bmod 12, \\ \left\lfloor \dfrac{1}{2} \right\rfloor & \text{if } k = 2 \bmod 12, \end{cases}$$

and

$$\dim S_k(\Gamma_1) = \dim M_k(\Gamma_1) - 1.$$

In particular, $\dim M_2(\Gamma_1) = 0$,

$$\dim M_4(\Gamma_1) = \dim M_6(\Gamma_1) = \dim M_8(\Gamma_1) = \dim M_{10}(\Gamma_1) = \dim M_{14}(\Gamma_1) = 1,$$

and $\dim M_{12}(\Gamma_1) = 2$.

Proof. We have $\dim M_2(\Gamma_1) = 0$ by the bound. The Eisenstein series E_k are nonzero and span $M_k(\Gamma_1)$ for $k = 4,\ 6,\ 8,\ 10,\ 14$ (which by the bound are at most 1-dimensional).

For the general dimension formula, we need to show $\dim M_{k+12} = \dim M_k + 1$. This follows from the cases $0 \leq k \leq 12$ and 5.1. □

Corollary 5.7 (Zagier's unreasonable effectiveness)**.** Two modular forms of the same weight for the same group coincide if their Fourier coefficients coincide up to N for N sufficiently large.

Lemma 5.8. $\Delta = \frac{1}{1728}\left(E_4^3 - E_6^2\right)$.

Expressing Δ in terms of E_6^2 and E_{12} gives $\tau(n) = \sigma_{11}(n) \bmod 691$.

Definition 5.9. Define the j-invariant

$$j(\tau) = \frac{E_4^3}{\Delta(\tau)} = 1728 \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}.$$

This is a meromorphic modular function of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$. We have

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

The minimal degree of a faithful complex representation of the monster group is 196883 (moonshine).

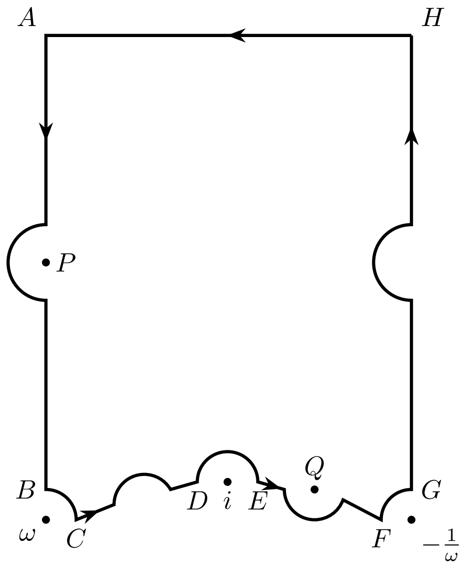
Proposition 5.10. The j-invariant defines a holomorphic bijection from $\mathrm{SL}_2(\mathbb{Z})\backslash\overline{\mathbb{H}}$ and the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

Proof. The j-invariant has a simple pole at ∞ and is holomorphic in $\mathbb{H}$ (weakly holomorphic). We apply

$$\mathrm{ord}_\infty(f) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\omega(f) + \sum_{\substack{P \in \Gamma_1\backslash\mathbb{H}, \\ P \neq i, \omega}} \mathrm{ord}_P(f) = \frac{k}{12}$$

to $f(\tau) = j(\tau) - c$ for $c \in \mathbb{C}$. Since $\mathrm{ord}_\infty(f) = -1$, there must be exactly one point P with $f(P) = 0$, that is, $j(P) = c$, hence a bijection. □

Proof of 5.4. We integrate $\frac{f'}{f}$ over the boundary of the fundamental domain $\mathcal{F}$. More precisely, we let $\mathcal{F}_T$ be the fundamental domain cut at height T, so that all poles and zeros of f are below iT (if that was not possible, f would not be a meromorphic function in q, i.e. at ∞). Then C is the contour which goes over the boundary of $\mathcal{F}_T$ with the following modifications. If a pole or zero lies at $\omega, -\frac{1}{\omega}, i$, these points are excluded. For other poles or zeros at the boundary, modify the contour such that each Γ_1-equivalence class of such points occurs exactly once in the interior of C.



By the residue theorem we have

$$\frac{1}{2\pi i}\int_C \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau = \sum_{\substack{P\in\Gamma_1\backslash\mathbb{H},\\ P\neq i,\omega}} \operatorname{ord}_P(f).$$

Now evaluate the integral term by term.

(1) The integral at the top gives $-\operatorname{ord}_\infty(f)$:

$$\frac{1}{2\pi i}\int_{\overline{HA}} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau = \frac{1}{2\pi i}\int_{-B_{e^{-2\pi T}}(0)} \frac{\partial f/\partial q}{f(q)}\,\mathrm{d}q = -\operatorname{ord}_\infty(f).$$

(2) The integral along the vertical sides cancel each other by periodicity.

(3) The integral along the arc around i gives $-\frac{1}{2}\operatorname{ord}_i(f)$. Take $a\in\mathbb{H}$, then the Laurent expansion of f is $a_m(\tau-a)^m+\cdots$. Then $\frac{f'(\tau)}{f(\tau)} = \frac{m}{\tau-a} + \text{holomorphic}$.

Integrating $\frac{f'}{f}$ counterclockwise over a circular arc of angle θ and radius ε gives $mi\theta = \operatorname{ord}_a(f)i\theta$ as $\varepsilon\to 0$. Here $\theta=\pi$ and we are integrating clockwise.

(4) The integral along the arcs around $\omega, -\frac{1}{\omega}$ give in the same fashion $-\frac{1}{6}\operatorname{ord}_\omega(f)$ (the angles are $\frac{\pi}{3}$).

Collecting all terms it remains to show

$$\frac{1}{2\pi i}\int_{\overline{CD}} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau + \frac{1}{2\pi i}\int_{\overline{EF}} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau \to \frac{k}{12} \qquad \varepsilon\to 0.$$

But now $S:\tau\mapsto -\frac{1}{\tau}$ takes $\overline{CD}$ to $-\overline{EF}=\overline{FE}$. Since

$$\frac{f'(\gamma\tau)}{f(\gamma\tau)}\,\mathrm{d}\gamma\tau = \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau + k\frac{c}{c\tau+d}\,\mathrm{d}\tau \qquad \gamma\in\Gamma_1,$$

we see

$$\int_C \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau - \int_{\gamma C} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau = \int_C -k\frac{c}{c\tau+d}\,\mathrm{d}\tau$$

for suitable contours C. Hence

$$\frac{1}{2\pi i}\int_{\overline{CD}} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau + \frac{1}{2\pi i}\int_{\overline{EF}} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau = -k\frac{1}{2\pi i}\int_{\overline{CD}} \frac{\mathrm{d}\tau}{\tau} \to \frac{k}{12}.$$

□

6. Tuesday 11 November 2025

Today throughout N will denote a positive integer.

Definition 6.1. Define the *principal congruence subgroup* of level N

$$\Gamma(N) \coloneqq \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod N\right\}.$$

In particular, $\Gamma(1)=\mathrm{SL}_2(\mathbb{Z})$.

Lemma 6.2. (1) $\Gamma(N)$ is the kernel of the natural homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
(2) The map is surjective, i.e. $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.
(3) The index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$ is then $|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$.

Definition 6.3. A *congruence subgroup* Γ of level N is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$. In particular, every congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$. (Not all subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index are congruence.)

Definition 6.4. Define the Hecke subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}.$$

and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

So we have $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$.

Lemma 6.5. (1) $\Gamma(N)$ is normal in $\Gamma_1(N)$ with $\Gamma_1(N)/\Gamma(N) = \mathbb{Z}/N\mathbb{Z}$.
(2) $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ with $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$.
(3) $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$.

Lemma 6.6. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be any subgroup (not necessarily of finite index). Let $\mathcal{F}$ be a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Then

$$\bigcup_{\gamma \in \overline{\Gamma} \backslash \mathrm{PSL}_2(\mathbb{Z})} \gamma \mathcal{F}$$

is a fundamental domain for Γ, where $\gamma \in \overline{\Gamma} \backslash \mathrm{PSL}_2(\mathbb{Z})$ runs over a set of right coset representative of $\mathrm{PSL}_2(\mathbb{Z})$ modulo Γ.

Example 6.7. The index of $\Gamma_0(2)$ in $\mathrm{SL}_2(\mathbb{Z})$ is 3 with right coset representatives given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then a fundamental domain for $\Gamma_0(2)$ looks as follows.

Note that $\Gamma_0(2)$ now has two (equivalence classes of) cusps.

Lemma 6.8. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N. Then Γ acts on the set of (rational) cusps $\{\infty\} \cup \mathbb{Q}$ with finitely many orbits. (Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$.) Often we call the Γ-equivalence classes of cusps just *cusps of* Γ.

Example 6.9. $\Gamma_0(p)$ has two cusps, ∞ and 0. $\Gamma_0(4)$ has three cusps, ∞, 0, $\frac{1}{2}$.

Lemma 6.10. $\Gamma_0(4)$ is generated by $\pm T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\pm ST^{-4}S = \mp \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$.

Definition 6.11 (Slash operator)**.** For f a function on $\mathbb{H}$, define

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau), \qquad \gamma \in \mathrm{SL}_2(\mathbb{R}).$$

This defines an action of the group $\mathrm{SL}_2(\mathbb{R})$ on the space of functions on $\mathbb{H}$,

$$(f|_k\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2.$$

So modularity of weight k for Γ means $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

Definition 6.12. A function f on $\mathbb{H}$ is *weakly modular* of level N for the congruence subgroup Γ of level N of weight k if $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

Definition 6.13. A weakly modular form f of level N is periodic with period N (since

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N) \subset \Gamma).$$

Hence f is a function of $q_N = e^{\frac{2\pi i\tau}{N}}$, that is, on the punctured disc around 0. Then we call f to be *meromorphic* at ∞ if it has the Laurent expansion (Fourier expansion)

$$f(\tau) = \sum_{n \gg -\infty} a_n q_N^n, \qquad \Im\tau \gg 0, \text{i.e. near } \infty.$$

We call f *holomorphic* at ∞ if $a_n = 0$ for $n < 0$.

Definition 6.14. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight k with respect to Γ if

(1) f is holomorphic on $\mathbb{H}$,
(2) f is weight-k invariant under Γ,
(3) $f|_k\alpha$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

If in addition $a_0 = 0$ in the Fourier expansion of $f|_k\alpha$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then f is a *cusp form* of weight k with respect to Γ.

Write $M_k(\Gamma)$ for the vector space of modular forms of weight k for Γ, $S_k(\Gamma)$ for cusp forms. If $\Gamma = \Gamma_0(N)$, we often replace Γ with N in the previous two notations.

Remark 6.15. (1) If $-I \notin \Gamma$, nonzero modular forms of odd weight may well exist, in contract to $\mathrm{SL}_2(\mathbb{Z})$.
(2) The condition on $f|_k\alpha$ at ∞ is also stated as being "holomorphic/vanishes at all cusps". The condition only depends on the Γ-equivalence classes of cusps and hence only needs to be checked for finitely many α.

Recall that from the exercises that we have defined

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \int_{\mathcal{F}_\Gamma} 1 \frac{\mathrm{d}u\mathrm{d}v}{v^2} =: \int_{\Gamma\backslash\mathbb{H}} 1 \frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

where $\mathcal{F}_\Gamma$ is (nice) fundamental domain of Γ. We also write $\mathrm{d}\mu(\tau)$ for the $\mathrm{SL}_2(\mathbb{R})$-invariant measure $\frac{\mathrm{d}u\mathrm{d}v}{v^2}$ on $\mathbb{H}$: $\mathrm{d}\mu(g\tau) = \mathrm{d}\mu(\tau)\ \forall g \in \mathrm{SL}_2(\mathbb{R})$.

Lemma 6.16. $\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}) = \frac{\pi}{3}$ and in general

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}) = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]\frac{\pi}{3}.$$

Proposition 6.17. Let f be a nonzero meromorphic modular form of weight k for a congruence subgroup Γ. Then

$$\sum_{P \in \Gamma\backslash\overline{\mathbb{H}}} \frac{1}{N_P} \operatorname{ord}(f) = \frac{k}{4\pi} \mathrm{vol}(\Gamma\backslash\mathbb{H}),$$

where N_P is the order of the stabiliser of the point $P \in \mathbb{H}$ in $\overline{\Gamma}$. For P a cusp, $N_P = 1$, and the order of f at P is measured in terms of the local variable q_h.

Proof. We first assume that k is even. Then considering if necessary the group generated by Γ and ± 1, we can assume that $-1 \in \Gamma$. (This does change either side of the asserted equation.) Then $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$ and representatives for $\overline{\Gamma}\backslash\mathrm{PSL}_2(\mathbb{Z})$ (which gives rise to a fundamental domain for Γ) lift to representatives for $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. Consider

$$F(\tau) := \prod_{\gamma \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})} (f|_k\gamma)(\tau).$$

F is a nonzero meromorphic form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]k$. We apply 5.4 to F.

If k is odd, consider $g(\tau) = f^2(\tau)$ which has weight $2k$. Then apply the above result and divide both sides of the equation by 2. □

Theorem 6.18. For a congruence subgroup Γ,

$$\dim M_k(\Gamma) \leq \frac{k}{4\pi} \mathrm{vol}(\Gamma\backslash\mathbb{H}) + 1.$$

Corollary 6.19. $M_k(\Gamma) = 0$ for $k < 0$ and $M_0(\Gamma) = \mathbb{C}$. Two modular forms for Γ coincide if their Fourier coefficients agree up to a high enough index.

We extend the slash operator to $\mathrm{GL}_2^+(\mathbb{Q})$ by

$$(f|_k\alpha)(\tau) := \det(\alpha)^{\frac{k}{2}} j(\alpha,\tau)^{-k} f(\alpha\tau).$$

In this way, the centre $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ acts trivially.

Example 6.20. (1) If $\Gamma' \subset \Gamma$, then $M_k(\Gamma) \subset M_k(\Gamma')$ and $S_k(\Gamma) \subset S_k(\Gamma')$.
(2) Let $d, N > 0$ and let $f \in M_k(\Gamma_0(N))$. Then

$$f(d\tau) = f|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} (\tau) \in M_k(\Gamma_0(dN)),$$

and similarly for cusps forms and $\Gamma_1(N)$.

Proposition 6.21. Let $f = \sum_{n=1}^{\infty} a_n q_N^n \in S_k(\Gamma)$ be a cusp form of weight k for a congruence subgroup Γ of level N. Then

(1) $(\Im\tau)^{\frac{k}{2}}|f(\tau)|$ is bounded on $\mathbb{H}$. (In fact, this is how one can define cusps forms without referring to Fourier expansions.)

(2) Hecke's bound: $|a_n| \leq Cn^{\frac{k}{2}}$ for some $C > 0$. (3.10 (Deligne) improves this to $O(n^{\frac{k-1}{2}})$.)

A Dirichlet character mod N is a (multiplicative) homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{C}^{\times}.$$

We extend χ to a function on $\mathbb{Z}/N\mathbb{Z}$ by setting $\chi(n) = 0$ for nonunit n mod N and then to a function on $\mathbb{Z}$ by $\chi(n) \coloneqq \chi(\overline{n})$. The smallest possible N for χ is the *conductor* of χ.

Definition 6.22 (Modular forms of Nebentypus)**.** For χ a Dirichlet character mod N, set

$$M_k(N,\chi) = \left\{ f \in M_k(\Gamma_1(N)) : f|_k\gamma = \chi(d)f \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}.$$

We have $M_k(N,\chi) = 0$ unless $\chi(-1) = (-1)^k$. We set $M_k(N,1) = M_k(N)$ for the trivial character mod N.

Lemma 6.23.

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi \bmod N} M_k(N,\chi).$$

Example 6.24. Let χ be the unique nontrivial Dirichlet character mod 4 given by $\chi(d) = (-1)^{\frac{d-1}{2}}$ for d odd. Then

$$M_k(\Gamma_1(4)) = \begin{cases} M_k(4,1) & \text{if } k \text{ is even,} \\ M_k(4,\chi) & \text{if } k \text{ is odd.} \end{cases}$$

7. Tuesday 18 November 2025

Definition 7.1 (Jacobi θ-series)**.** We define the *Jacobi θ-series/function*

$$\theta(\tau) \coloneqq \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

Proposition 7.2 (θ-transformation formula)**.**

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{-2i\tau}\,\theta(\tau).$$

Corollary 7.3. For the generators of $\overline{\Gamma_0(4)}$, (images of) $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $-ST^{-4}S = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, we have

$$\theta(T\tau) = \theta(\tau+1) = \theta(\tau)$$

and

$$\theta((-ST^{-4}S)\tau) = \theta\left(\frac{\tau}{4\tau+1}\right) = \sqrt{4\tau+1}\,\theta(\tau) = \sqrt{j((-ST^{-4}S),\tau)}\,\theta(\tau).$$

Note that for the mth power of the Jacobi theta function we have

$$\theta^m(\tau) = \sum_{n=0}^{\infty} r_m(n) q^n,$$

where

$$r_m(n) = \left| \left\{ (x_1,\ldots,x_m) \in \mathbb{Z}^m : \sum_{i=1}^{m} x_i^2 = n \right\} \right|$$

is the representation number of n as the sum of m squares. We immediately obtain

Corollary 7.4. Let $\chi(d) = \left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}}$ be the unique non-trivial Dirichlet character modulo 4. Then

$$\theta^{2k}(\tau) \in M_k(\Gamma_1(4)) = \begin{cases} M_k(4) & \text{if } k \text{ is even,} \\ M_k(4,\chi) & \text{if } k \text{ is odd.} \end{cases}$$

Proof. From the formulas at the top, it is clear that the desired transformation property holds for the generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mp ST^{-4}S = \pm \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ of $\Gamma_0(4)$. Hence it holds for the whole group by the cocycle relation. □

Proof of 7.2. Both sides are holomorphic in $\tau \in \mathbb{H}$, so it is enough to show this for $\tau = it$, that is,

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{2t}} = \sqrt{2t} \sum_{n\in\mathbb{Z}} e^{-2\pi t n^2}.$$

We will use the Poisson summation formula. It is well-known that $e^{-\pi x^2}$ is invariant under Fourier transform:

$$\widehat{e^{-\pi x^2}}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x y}\, \mathrm{d}x = e^{-\pi y^2}.$$

A simple change of variables then yields

$$\widehat{e^{-\frac{\pi x^2}{2t}}}(y) = \sqrt{2t} e^{-2\pi t y^2} \qquad \forall t > 0.$$

Then Poisson summation

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad \text{for a suitable } f$$

gives the result. □

Proof of 7.3. The transformation $\tau \mapsto \frac{\tau}{4\tau+1}$ is given by $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, whereas the θ-transformation formula is in terms of the Fricke involution $\omega_4 = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$. Since $-4 \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \omega_4 T^{-1} \omega_4$, we have

$$\theta\left(\frac{\tau}{4\tau+1}\right) = \theta(\omega_4 T^{-1} \omega_4 \tau).$$

Setting for the moment $f(\tau) = \sqrt{-2i\tau}$, we use the θ-transformation formula to obtain

$$\theta\left(\frac{\tau}{4\tau+1}\right) = \theta(\omega_4 T^{-1}\omega_4\tau) = f(T^{-1}\omega_4\tau)\theta(T^{-1}\omega_4\tau) = f(T^{-1}\omega_4\tau)\theta(\omega_4\tau) = f(T^{-1}\omega_4\tau)f(\tau)\theta(\tau).$$

But

$$f(T^{-1}\omega_4\tau)f(\tau) = \sqrt{2i\left(\frac{1}{4\tau}+1\right)}\sqrt{-2i\tau} = \sqrt{4\tau+1},$$

where the last equality is not trivial. □

Theorem 7.5. For the representation numbers $r_m(n)$ of n as the sum of an even number of squares, we have

$$r_2(n) = 4 \sum_{d|n} (-1)^{\frac{d-1}{2}}$$

and

$$r_4(n) = 8(2 + (-1)^n) \sum_{\substack{d|n,\\ d \text{ odd}}} d = \begin{cases} 8\sigma_1(n) & \text{if } n \text{ is odd,} \\ 24\sigma_1(n_0) & \text{if } n = 2^r n_0 \text{ with } 2 \nmid n_0. \end{cases}$$

In particular, every positive integer is the sum of 4 squares (Lagrange). Moreover,

$$r_6(n) = 16 \sum_{\substack{d|n,\\ \frac{n}{d} \text{ odd}}} (-1)^{\frac{\frac{n}{d}-1}{2}} d^2 - 4 \sum_{\substack{d|n,\\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} d^2$$

and

$$r_8(n) = \begin{cases} 16\sigma_3(n) & \text{if } n \text{ is odd,} \\ 16\sigma_3(n) - 32\sigma_3\left(\frac{n}{2}\right) & \text{if } n = 2 \bmod 4, \\ 16\sigma_3(n) - 32\sigma_3\left(\frac{n}{2}\right) + 256\sigma_3\left(\frac{n}{4}\right) & \text{if } n = 0 \bmod 4. \end{cases}$$

For $m = 0 \bmod 4$ with $m \geq 12$, we have

$$r_m(n) = -\frac{(-1)^{\frac{m}{4}}}{2^{\frac{m}{2}} - 1} \frac{m}{B_{\frac{m}{2}}} \sigma_{\frac{m}{2}-1}(n) + O\left(n^{\frac{m}{4}}\right) \qquad \text{if } n \text{ is odd.}$$

Here $B_{\frac{m}{2}}$ is the Bernoulli number. For n even, the proof gives a formula similar to $r_6(n)$ and $r_8(n)$ (with error term). For $m = 2 \bmod 4$ with $m \geq 10$, there is a formula similar to $r_6(n)$ involving twisted power divisor sums with exponent $\frac{m}{2} - 1$, again with an error term $O\left(n^{\frac{m}{4}}\right)$.

Proof. The basic idea is to write $\theta^m(\tau)$ as an explicit linear combination of Eisenstein series and a cusp form. But $S_k(\Gamma_1(4)) = 0$ for $k = 1, 2, 3, 4$ (verify). Moreover, we have formulas for the (very small) dimensions of $M_k(\Gamma_1(4))$ in this range. For $m = 4$, we compare Fourier coefficients to obtain

$$\theta^4 = \frac{1}{3}\left(4E_2(4\tau) - E_2(\tau)\right),$$

from which the formula for $r_4(m)$ follows.

For $m = 8$, $\theta^8(\tau)$ is a linear combination of $E_4(\tau)$, $E_4(2\tau)$, and $E_4(4\tau)$. Explicitly, we have by comparing coefficients

$$\theta^8(\tau) = \frac{1}{15}E_4(\tau) - \frac{2}{15}E_4(2\tau) + \frac{16}{15}E_4(4\tau).$$

For $m \geq 12$ with $m = 0 \bmod 4$, $\theta^m(\tau)$ is a cusp form up to a linear combination of $E_{\frac{m}{2}}(\tau)$, $E_{\frac{m}{2}}(2\tau)$, and $E_{\frac{m}{2}}(4\tau)$. More precisely (verify this), with $k = \frac{m}{2}$ we have

$$\theta^m(\tau) - \frac{1}{2^k - 1}\left[(-1)^{\frac{k}{2}}E_k(\tau) - \left(1 + (-1)^{\frac{k}{2}}\right)E_k(2\tau) + 2^k E_k(4\tau)\right] \in S_k(\Gamma_0(4)).$$

The claim now follows from Hecke's bound for cusp forms.

The case $m = 2$ goes through as above once one knows that

$$1 + 4\sum_{n=1}^{\infty}\sum_{d|n}(-1)^{\frac{d-1}{2}}q^n$$

is a modular form. We will show this later using Hecke operators. □

Remark 7.6. We have

$$n^{k-1} \leq \sigma_{k-1}(n) = n^{k-1}\sum_{d|n} d^{1-k} < n^{k-1}\zeta(k-1) \qquad \text{if } k \geq 3.$$

Hence $\sigma_{k-1}(n) = O(n^{k-1})$ for $k \geq 3$. For $k = 1, 2$, note that we can bound $d(n) = \sigma_0(n)$, the number of divisors of n, by $C_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$ and a constant only depending on ε. Hence in that case we have $O(n^{k-1+\varepsilon})$ for all $\varepsilon > 0$. In conclusion, the representation numbers of sums of squares grow roughly like n^{k-1}.

Remark 7.7 (Modular forms of half-integral weight)**.** $\theta(\tau)$ is some kind of modular form for $\Gamma_0(4)$ of weight $\frac{1}{2}$. However, one needs to be careful. Because of the square-root, one has for $\gamma \in \Gamma_0(4)$ only

$$\theta(\gamma\tau) = \psi(\gamma)\sqrt{j(\gamma,\tau)}\,\theta(\tau),$$

where $\psi(\gamma)$ is a 4th root of unity. ψ is not quite a multiplicative character on $\Gamma_0(4)$. For example, for $\gamma = -1$, we must have $\psi(-1) = -i$. Furthermore, the cocycle relation does not hold for $\sqrt{j(\gamma,\tau)}$, only up to sign (verify). One can show

$$\psi(\gamma) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}, \qquad \varepsilon_d := \begin{cases} 1 & \text{if } d = 1 \bmod 4, \\ i & \text{if } d = 3 \bmod 4, \end{cases}$$

for $\gamma \in \Gamma_0(4)$. (This is nontrivial!). One calls $\psi(g)$ the "theta multiplier".

In any case, even without knowing this formula, one can *define* an automorphy factor by

$$J(\gamma,\tau) := \frac{\theta(\gamma\tau)}{\theta(\tau)},$$

which (trivially) satisfies the cocycle relation (verify). One then defines modular forms of half-integral weight for congruence subgroups of $\Gamma_0(4)$ with respect to $J(\gamma,\tau)$. Namely, a function f holomorphic on $\mathbb{H}$ and at the cusps of a congruence subgroup $\Gamma \subset \Gamma_0(4)$ is called a *modular form of half-integral weight* $\frac{k}{2}$ for k odd if

$$f(\gamma\tau) = J(\gamma,\tau)^k f(\tau) \qquad \forall\gamma \in \Gamma.$$

Note that for this definition one does *not* need to know what $J(\gamma,\tau)$ exactly is. However, one easily sees a priori

$$J^2(\gamma,\tau) = (-1)^{\frac{d-1}{2}}(c\tau + d).$$

Indeed, this holds for the generators of $\Gamma_0(4)$ by 7.2 and then for all $\gamma \in \Gamma_0(4)$ by the cocycle relation for the right hand side.

Remark 7.8 (Reminders from linear algebra)**.** (1) A symmetric $m \times m$ matrix S gives rise to a bilinear form on $\mathbb{R}^m$ via $(x,y) = {}^t x S y$. The associated quadratic form is $Q(x) = \frac{1}{2} {}^t x S x$.

(2) S is *positive definite* if $Q(x) > 0$ for $x \neq 0$.
(3) S is *even* if S is integral with even entries on the diagonal.
(4) The *level* of S the smallest $N \in \mathbb{N}\backslash\{0\}$ such that NS^{-1} is even.
(5) S and S' are *equivalent* if $S' = {}^t ASA =: S[A]$ for some $A \in \mathrm{GL}_m(\mathbb{Z})$.
(6) There are finitely many equivalence classes of given discriminant.
(7) Examples:
   (a) $Q(x) = \sum_i x_i^2$, $S = \mathrm{diag}\{2, \ldots, 2\}$ with level 4, determinant 2^m.
   (b) $Q(x) = 3x_1^2 + x_1 x_2 + x_2^2$, $S = \begin{pmatrix} 6 & 1 \\ 1 & 2 \end{pmatrix}$, with level and determinant 11.

Definition 7.9. Let S be positive definite. Define

$$\theta(\tau, S) = \sum_{x \in \mathbb{Z}^m} e^{\pi i S[x] \tau} = \sum_{x \in \mathbb{Z}^m} q^{Q(x)},$$

where $S[x] = {}^t x S x$. Since S is positive definite, the series converges absolutely and uniformly in bounded subsets (verify). Hence $\theta(\tau, S)$ is a holomorphic function on $\mathbb{H}$. An equivalent S has the same theta series.

Remark 7.10 (Generating series of representation numbers)**.** For S even, we have $Q(x) \in \mathbb{Z}$ for all x, so

$$\theta(\tau, S) = \sum_{n=0}^{\infty} r_S(n) q^n$$

where

$$r_S(n) = |\{x \in \mathbb{Z}^m : S[x] = n\}|$$

is the representation number of n by S (which is finite; verify).

Proposition 7.11 (General θ-transformation formula)**.**

$$\theta\left(-\frac{1}{\tau}, S\right) = \frac{\sqrt{-i\tau}^m}{\sqrt{\det S}} \theta\left(\tau, S^{-1}\right).$$

Proof. Similarly to the proof for the Jacobi one: both sides are holomorphic in $\tau \in \mathbb{H}$, so it is enough to show this for $\tau = it$, that is,

$$\sum_{x \in \mathbb{Z}^m} e^{-\frac{\pi S[x]}{t}} = \frac{\sqrt{t}^m}{\sqrt{\det S}} \sum_{x \in \mathbb{Z}^m} e^{-\pi t S^{-1}[x]}.$$

We will use the m-dimensional Poisson summation formula. It is well-known that $e^{-\pi \|x\|^2}$ where $\|x\|^2 = {}^t x x$ is invariant under Fourier transform for $\mathbb{R}^m$:

$$\widehat{e^{-\pi\|x\|^2}}(y) = \int_{\mathbb{R}^m} e^{-\pi \|x\|^2} e^{-2\pi i ({}^t x y)} \mathrm{d}x = e^{-\pi \|y\|^2}.$$

In fact, this follows immediately from the 1-dimensional case. A simple change of variables and the general transformation/substitution rule then yields

$$\widehat{e^{-\frac{\pi\|x\|^2}{t}}}(y) = \frac{\sqrt{t}^m}{\sqrt{\det S}} e^{-\pi t \|y\|^2} \qquad \forall t > 0.$$

Then m-dimensional Poisson summation

$$\sum_{x \in \mathbb{Z}^m} f(x) = \sum_{x \in \mathbb{Z}^m} \widehat{f}(x) \qquad \text{for suitable } f$$

gives the result. □

Corollary 7.12. Let S be an even positive definite of rank m and suppose S is unimodular: $S = S^{-1}$. Then $m = 0 \bmod 8$ and

$$\theta(\tau + 1, S) = \theta(\tau, S), \qquad \theta\left(-\frac{1}{\tau}, S\right) = \tau^{\frac{m}{2}} \theta(\tau, S),$$

that is,

$$\theta(\theta, S) \in M_{\frac{m}{2}}\left(\mathrm{SL}_2(\mathbb{Z})\right).$$

Proof. It's clear that if $m = 0 \bmod 8$, then $\theta(\tau, S) \in M_{\frac{m}{2}}(\mathrm{SL}_2(\mathbb{Z}))$.

To show $m = 0 \bmod 8$, we first assume $m = 2k$ is even. Use the identity $(ST)^3 = -I$ so that $\theta((ST)^3\tau) = \theta(\tau)$. First set $f(\tau) = (-\tau)^k$. Then using 7.11 and $\theta(T\tau) = \theta(\tau)$ (since S is even), we calculate

$$\begin{aligned}\theta(\tau) = \theta((ST)^3\tau) &= f(TSTST\tau)\theta(TSTST\tau) = f(TSTST\tau)f(TST\tau)\theta(TST\tau)\\ &= f(TSTST\tau)f(TST\tau)f(T\tau)\theta(\tau).\end{aligned}$$

Now

$$f(TSTST\tau)f(TST\tau)f(\tau) = \left(-i\left(-\frac{1}{\tau}\right)\right)^k \left(-i\left(\frac{\tau}{\tau+1}\right)\right)^k (-i(\tau+1))^k = i^{3k} = (-i)^k.$$

But this is equal to 1 if and only if $k = 0 \bmod 4$, that is, $m = 0 \bmod 8$.

If m is odd, run the above argument for the unimodular matrix $S \oplus S$ of rank $2m$. ☐

Lemma 7.13. Let S be even positive definite of level N and (not necessarily even) rank m. Then

$$\theta(T\tau, S) = \theta(\tau, S), \qquad \theta\left(\begin{pmatrix}1 & 0\\ N & 1\end{pmatrix}\tau, S\right) = \sqrt{N\tau+1}^{\,m}\theta(\tau, S).$$

Proof. First note that $-ST^{-N}S = \begin{pmatrix}1 & 0\\ N & 1\end{pmatrix} \in \Gamma_0(N)$. Then the proof goes (almost) exactly the same way as 7.3; use that $\theta(\tau, S^{-1})$ is N-periodic. ☐

This suggests that $\theta(\tau, S)$ is a modular form of weight $\frac{m}{2}$ for $\Gamma_0(N)$. However, it does not prove it, since $T, \pm 1$ and $\begin{pmatrix}1 & 0\\ N & 1\end{pmatrix}$ do not generate $\Gamma_0(N)$ in general, unlike for $\Gamma_0(4)$. Nevertheless we have the following.

Theorem 7.14 (Hecke–Schöneberg)**.** Let S be even and of even rank $m = 2k$, level N and discriminant $D := (-1)^k \det S$. Let $\chi_D = \left(\frac{D}{-}\right)$ be the associated Dirichlet character mod N. Then

$$\theta(\tau, S) \in M_k(N, \chi_D).$$

In particular, if S is unimodular, $\theta(\tau, S) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Proposition 7.15. Let S be even, unimodular, positive definite and of rank m. For the representation number $r_S(n)$ we have

$$r_S(n) = -\frac{2k}{B_k}\sigma_{k-1}(n) + O\left(n^{\frac{k}{2}}\right) \qquad n \to \infty,$$

where B_k denotes the kth Bernoulli number (which is negative for $k = 0 \bmod 4$). 3.10 improves the bound to $O\left(n^{\frac{k-1}{2}}\right)$.

Proof. We can write

$$\theta(\tau, S) = E_k(\tau) + f(\tau) \qquad \text{with } f \in S_k(\mathrm{SL}_2(\mathbb{Z})) \text{ a cusp form.}$$

Hecke's bound and the second equality of 3.16 gives the claim. ☐

Example 7.16 (Three quaternary forms of level 11)**.** Recall from the overview lecture the three even positive definite quadratic forms of level 11 and discriminant 11^2 given by

$$\begin{aligned}P(x,y,u,v) &= x^2 + xy + 3y^2 + u^2 + uv + 3v^2\\ Q(x,y,u,v) &= 2(x^2+y^2+u^2+v^2) + 2xu + xv + yu - 2yv\\ R(x,y,u,v) &= x^2 + 4(y^2+u^2+v^2) + xu + 4yu + 3yv + 7uv\end{aligned}$$

with Gram matrices

$$S = \begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 6 & 0 & 0\\ 0 & 0 & 2 & 1\\ 0 & 0 & 1 & 6\end{pmatrix}, \quad T = \begin{pmatrix}4 & 0 & 2 & 1\\ 0 & 4 & 1 & -2\\ 2 & 1 & 4 & 0\\ 1 & -2 & 0 & 4\end{pmatrix}, \quad U = \begin{pmatrix}2 & 0 & 1 & 0\\ 0 & 8 & 4 & 3\\ 1 & 4 & 8 & 7\\ 0 & 3 & 7 & 8\end{pmatrix}, .$$

We have

$$\theta(\tau, S), \theta(\tau, T), \theta(\tau, U) \in M_2(11)$$

with

$$\theta(\tau, S) = 1 + 4q + 4q^2 + \cdots, \quad \theta(\tau, T) = 1 + 0q^2 + 12q^2 + \cdots, \quad \theta(\tau, U) = 1 + 6q + q^2 + \cdots$$

The value of all three θ-series at the cusp 0 is given by $-\frac{1}{11}$ by the θ-transformation formula.

Our dimension estimate gives $\dim M_2(11) \leq 3$. Indeed, the "total vanishing degree" for a nonzero f is $\frac{k[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(11)]}{12} = \frac{2\cdot 12}{12} = 2$. In fact, we have $\dim M_2(11) = 2$, since $\dim S_2(11) \leq 1$ (two cusps; soon we see in fact $\dim S_2(11) = 1$) and $\dim \mathcal{E}_2(11) = 1$ (while there are two cusps, we have only one holomorphic series due to the convergence issue for weight 2).

We do have $\dim S_2(11) = 1$:

$$\theta(\tau, S) - \theta(\tau, T) = 4q - 8q^2 + \cdots \in S_2(11).$$

But also

$$\theta(\tau, T) - \theta(\tau, U) = -6q + 12q^2 + \cdots \in S_2(11).$$

So

$$\frac{3}{2}\left(\theta(\tau, S) - \theta(\tau, T)\right) = \theta(\tau, T) - \theta(\tau, U).$$

That is, we proved

$$r_U(n) = \frac{3}{2} r_S(n) - \frac{1}{2} r_T(n) \qquad \forall n \neq 0.$$

We had also shown conditionally on $\dim S_2(11) \neq 0$ (which we now have)

$$(\eta(\tau)\eta(11\tau))^2 = q\prod_{n=1}^{\infty}(1-q^n)^2\left(1-q^{11n}\right)^2 = q - 2q^2 + \cdots \in S_2(11),$$

so

$$4\left(\eta(\tau)\eta(11\tau)\right)^2 = \theta(\tau, T) - \theta(\tau, U).$$

The Eisenstein series is given by

$$\begin{aligned} F(\tau) &:= -\frac{1}{24}\left(E_2(\tau) - 11E_2(11\tau)\right) = \frac{5}{12} + \sum_{n=1}^{\infty}\left(\sigma(n) - 11\sigma\left(\frac{n}{11}\right)\right)q^n \\ &= \frac{5}{12} + q + 3q^2 + \cdots \in M_2(11), \end{aligned}$$

so checking the first two coefficients we see

$$\theta(\tau, S) = \frac{12}{5}F(\tau) + \frac{8}{5}\left(\eta(\tau)\eta(11\tau)\right)^2.$$

Hence using 3.10 for the coefficients of weight 2 cusp forms, we obtain

$$\left|r_S(n) - \frac{12}{5}\sigma(n)\right| \leq \frac{8}{5}\sigma_0(n)n^{\frac{1}{2}} \qquad (n, 11) = 1,$$

where $\sigma_0(n) = d(n)$ denotes the number of positive divisors of n.

One has $\sigma_0(n) = O\left(n^{\varepsilon}\right)$ and $\sigma(n) = O\left(n^{1+\varepsilon}\right)$ for all $\varepsilon > 0$.

In fact, using

$$-\frac{1}{24}\left(E_2(\tau) - pE_2(p\tau)\right) = \frac{p-1}{24} + \sum_{n=1}^{\infty}\left(\sigma(n) - p\sigma\left(\frac{n}{p}\right)\right)q^n = \frac{5}{12} + q + 3q^2 + \cdots$$

is the only Eisenstein series in $M_2(p)$ for any prime p, we have the following.

Proposition 7.17. Let S be even positive definite or rank 4, level p, and discriminant p^2. Then

$$r_S(n) = \frac{24}{p-1}\sigma(n) + O\left(n^{\frac{1}{2}+\varepsilon}\right) \qquad (n, p) = 1.$$

Remark 7.18. We have encountered two important problems in the theory of positive definite quadratic forms for which the theory of modular forms is particularly suited:

(1) Find linear relations between representation numbers.
(2) Give (asymptotic formulas) for the representation numbers.

Example 7.19 (E_8-lattice and other unimodular lattices)**.** (1) Let $L = \Lambda_8$ be the lattice of rank 8 inside $\mathbb{R}^8$ spanned by $u_i = e_i - e_{i+1}$ for $1 \leq i \leq 6$, $u_7 = e_6 + e_7$ and $u_8 = -\frac{1}{2}(e_1 + \cdots + e_8)$. Then $(u_i, u_i) = 2$ and $(u_i, u_j) = 0$ or -1. This gives rise to the E_8-Dynkin diagram in the theory of Lie algebras. Define S by $s_ij = (u_i, u_j)$, then S is unimodular (verify). Since $M_4(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_4$, we see

$$\theta(\tau, S) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n.$$

In particular, for all $m = 0 \bmod 8$, we have positive definite even unimodular lattices.

(2) In dimension 16, there are two classes of inequivalent lattices: $\Lambda_8 \oplus \Lambda_8$ and Λ_{16} (which is indecomposable). But since $M_8(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_8$, they have the same θ-series, so you cannot distinguish the two lattices by their representation numbers.
(3) In dimension 24, there are 24 classes of inequivalent lattices. One of them is the famous Leech lattice which has no vectors x of length $Q(x) = 1$. This gives another proof of the congruence mod 691 of the Ramanujan τ-function.
(4) In dimension 32 and higher, the classification is still open (and very difficult: for 32 there are at least 80 million classes). (This kind of classification problems can be attacked by the Siegel mass formula).

8. Tuesday 25 November 2025

Definition 8.1. For a complex variable s, define the *Riemann ζ-function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \Re s > 1$$

which converges absolutely and uniformly in vertical stripe in the region $\Re s > 1$. Hence $\zeta(s)$ defines a holomorphic function in this region.

Lemma 8.2. The Riemann ζ-function has the following Euler product.

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}. \qquad \Re s > 1$$

where the product extends over all prime p of $\mathbb{N}$.

Proof. We write

$$\prod_p \frac{1}{1-p^{-s}} = \prod_p \sum_{k=0}^{\infty} p^{-ks}.$$

Multiplying out the product now shows that by the fundamental theorem of arithmetic that every summand n^{-s} for $n > 0$ occurs exactly once.

(Really just a sketch, e.g., convergence of infinite products: By definition when the series of the logarithm of the terms converges). □

Corollary 8.3. The Riemann ζ-function does not vanish for $\Re s > 1$.

Definition 8.4. Define the *completed Riemann ζ-function*

$$Z(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Here $\Gamma(s)$ is the Γ-function which is a meromorphic function on $\mathbb{C}$ and for $\Re s > 0$ is given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t}.$$

We call $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$ the zeta factor at ∞ ("infinite prime") and write $L_\infty(s)$. We also write

$$L_p(s) = (1-p^{-s})^{-1}$$

for the zeta/L-factors at the finite primes so that

$$Z(s) = \prod_{p \leq \infty} L_p(s).$$

Theorem 8.5. The completed Riemann ζ-function has a meromorphic continuation to the whole complex $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$ with residue -1 and 1 respectively. Furthermore, it satisfies the functional equation

$$Z(1-s) = Z(s),$$

that is,

$$\pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Corollary 8.6. $\zeta(s)$ has a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$ with residue 1. It satisfies the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos\frac{\pi s}{2} \zeta(s).$$

Remark 8.7 (Some facts about the Γ-function)**.** (1) The Γ-function has a meromorphic continuation to $\mathbb{C}$. We have $\Gamma(s) \neq 0$ for all s and it has simple poles at $s = -n$ with $n = 0, 1, 2, 3, \ldots$ with residue $\frac{(-1)^n}{n!}$.
(2) $\Gamma(s+1) = s\Gamma(s)$.
(3) $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$.
(4) $\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s)$.
(5) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \Gamma(1) = 1,\ \Gamma(k+1) = k!$.

Proof of 8.6. We have $\zeta(s) = \pi^{\frac{s}{2}} \frac{Z(s)}{\Gamma\left(\frac{s}{2}\right)}$. This gives the analytic continuation with no additional poles. $Z(s)$ has a pole at $s = 0$, but so has $\Gamma\left(\frac{s}{2}\right)$, hence $\zeta(s)$ has no pole at $s = 0$. At $s = 1$, $Z(s)$ has a pole, so has $\zeta(s)$, since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, which gives residue 1 too. We have

$$\zeta(1-s) = \pi^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s),$$

from which the functional equation follows using the functional equations for Γ. □

Remark 8.8. (1) $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$.
(2) $\zeta(1-2k) = -\frac{B_{2k}}{2k}$ for $k \in \mathbb{N}\backslash\{0\}$.
(3) $\zeta(2k) = -\frac{(-1)^k (2\pi)^{2k}}{2(2k!)} B_{2k}$.
(4) $\zeta(2k+1) =$? open; $\zeta(3)$ is known to be irrational.
(5) $\zeta(-2k) = 0$, trivial zeros, which by the functional equation are the only zeros for $\Re s < 0$.
(6) All other zeros lie in the critical strip $0 \leq \Re s \leq 1$. The Riemann hypothesis (1859) proposes that all zeros in the critical strip are located on the critical line $\Re s = \frac{1}{2}$. We know $\zeta(s) \neq 0$ for $\Re s = 1$. This has already significant arithmetic consequences: it is equivalent to the prime number theorem.
(7) While the first trillion (10^{13}) zeros are indeed on the critical line, one has not been able to shrink the critical strip to $\varepsilon \leq \Re s \leq 1 - \varepsilon$ for any $\varepsilon > 0$.

Lemma 8.9. Set

$$\vartheta(\tau) = \theta\left(\frac{\tau}{2}\right) = 1 + 2\sum_{n=1}^{\infty} e^{\pi i \tau n^2}.$$

Then

$$\vartheta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\vartheta(\tau).$$

Proposition 8.10. Define the Mellin transform of $\vartheta(\tau)$

$$\Lambda(\vartheta, s) = \int_0^\infty (\vartheta(it) - 1) t^s \frac{\mathrm{d}t}{t} \qquad \Re s > \frac{1}{2}.$$

Then $\Lambda(\vartheta, s)$ converges absolutely and uniformly in compact subsets for $\Re s > \frac{1}{2}$ and therefore defines a holomorphic function in that region. Moreover,

$$Z(2s) = \frac{1}{2}\Lambda(\vartheta, s) \qquad \Re s > \frac{1}{2}.$$

Proof. $\vartheta(it) - 1$ is rapidly decreasing as $t \to \infty$, hence the integration is well defined. At $t = 0$, we have

$$(\vartheta(it) - 1)t^{s-1} = \left(t^{-\frac{1}{2}}\vartheta\left(\frac{i}{t}\right) - 1\right)t^{s-1} = t^{\frac{s-3}{2}}\vartheta\left(\frac{i}{t}\right) + t^{s-1},$$

which is integrable at $t = 0$ for $\Re s > \frac{1}{2}$. Finally,

$$\frac{1}{2}\Lambda(\vartheta, s) = \int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^s \frac{\mathrm{d}t}{t} = \sum_{n=1}^{\infty} (\pi n^2)^{-s} \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t} = \pi^{-s}\Gamma(s)\zeta(2s) = Z(2s).$$

Here both sides converge absolutely for $\Re s > \frac{1}{2}$, hence interchange of integration and summation is allowed. □

Proof of 8.5. We first split the interval of integration $(0, \infty)$ for the Mellin transformation into $(0, 1)$ and $(1, \infty)$:

$$2Z(2s) = \Lambda(\vartheta, s) = \int_0^1 (\vartheta(it) - 1)t^s \frac{\mathrm{d}t}{t} + \int_1^\infty (\vartheta(it) - 1)t^s \frac{\mathrm{d}t}{t}.$$

The second integral is defined for all $s \in \mathbb{C}$, while the first for $\Re s > \frac{1}{2}$. For the first integral, we change variables $t \mapsto \frac{1}{t}$ and obtain

$$\int_0^1 (\vartheta(it)-1)t^s \frac{\mathrm{d}t}{t} = \int_1^\infty \int_0^1 \left(\vartheta\left(\frac{i}{t}\right)-1\right) t^{-s}\frac{\mathrm{d}t}{t}.$$

Using 8.9 we get (a priori for $\Re s > \frac{1}{2}$)

$$\int_1^\infty \left(t^{\frac{1}{2}}\vartheta(it)-1\right) t^{-s}\frac{\mathrm{d}t}{t} = \int_1^\infty (\vartheta(it)-1)\, t^{\frac{1}{2}-s}\frac{\mathrm{d}t}{t} - \int_1^\infty t^{-s}\frac{\mathrm{d}t}{t} + \int_1^\infty t^{\frac{1}{2}-s}\frac{\mathrm{d}t}{t}.$$

All individual integrals converge for $\Re s > 1/2$, so the calculation is valid. Evaluating the last two integrals and combining this with the second integral, we have

$$2Z(2s) = \int_1^\infty (\vartheta(it)-1)\, t^2 \frac{\mathrm{d}t}{t} + \int_1^\infty (\vartheta(it)-1)t^{\frac{1}{2}-s}\frac{\mathrm{d}t}{t} - \frac{1}{s} + \frac{1}{s-\frac{1}{2}}.$$

But since $\vartheta(it)-1$ is rapidly decreasing, the above now converges absolutely for all $s \in \mathbb{C}$ except at $s=0$ and $s=\frac{1}{2}$. This gives the analytic continuation together with the poles and their residues. Moreover, the above is clearly invariant under the substitution $s \mapsto \frac{1}{2}-s$. So

$$Z(2s) = Z\left(2\left(\frac{1}{2}-s\right)\right) = Z(1-2s).$$

□

Definition 8.11. For a continuous function $F : \mathbb{R}_+ \to \mathbb{C}$ with $\lim_{t\to\infty} F(t) = a$, define its *Mellin transform* by

$$\Lambda(F,s) = \int_0^\infty (F(t)-a)t^s \frac{\mathrm{d}t}{t},$$

provided the integral exists.

Theorem 8.12 (Mellin principle)**.** Let $F, G : \mathbb{R}_+ \to \mathbb{C}$ be continuous functions such that

$$F(t) = a_0 + O\left(e^{-ct^\alpha}\right), \qquad G(t) = b_0 + O\left(e^{-ct^\alpha}\right) \qquad t \to \infty$$

for some constants $c, \alpha > 0$. Suppose

$$F\left(\frac{1}{t}\right) = Ct^k G(t)$$

for some real $k > 0$ and $0 \neq C \in \mathbb{C}$. Then

(1) The integrals defining the Mellin transforms $\Lambda(F,s)$ and $\Lambda(G,s)$ are absolutely and uniformly convergent in bounded subsets of the region defined by $Res > k$ and define holomorphic functions in this region.
(2) $\Lambda(F,s)$ and $\Lambda(G,s)$ have an analytic continuation to $\mathbb{C}\backslash\{0,k\}$ with potentially simple poles at $s=0$ and $s=k$ with residues

$$\mathrm{Res}_{s=0}\,\Lambda(F,s) = -a_0, \qquad \mathrm{Res}_{s=k}\,\Lambda(F,s) = Cb_0,$$
$$\mathrm{Res}_{s=0}\,\Lambda(G,s) = -b_0, \qquad \mathrm{Res}_{s=k}\,\Lambda(G,s) = C^{-1}a_0.$$

(3) We have the functional equation

$$\Lambda(F,s) = C\Lambda(G,k-s).$$

Proof (sketch). One proceeds in exactly the same way as before.

(1) The integral defining the Mellin transforms converges absolutely since at ∞ we have $F(t)-a_0, G(t)-b_0$ are rapidly decreasing, while at $t=0$, using the functional equation, we see that the integrands are $O\left(t^{s-k-1}\right)$ as $t \to 0$.
(2)(3) Splitting the interval of integration $(0,\infty)$ for the Mellin transform into $(0,1)$ and $(1,\infty)$ and using the functional equation we obtain as before

$$(*) \qquad \Lambda(F,s) = \int_1^\infty (F(t)-a_0)t^s\frac{\mathrm{d}t}{t} + C\int_1^\infty (G(t)-b_0)t^{k-s}\frac{\mathrm{d}t}{t} - \frac{a_0}{s} + \frac{Cb_0}{s-k}.$$

This gives the meromorphic continuation of $\Lambda(F,s)$ with the description of the simple poles. Switching F and G one sees

$$(\dagger) \qquad \Lambda(G,s) = \int_1^\infty (G(t)-b_0)t^s\frac{\mathrm{d}t}{t} + \frac{1}{C}\int_1^\infty (F(t)-a_0)t^{k-s}\frac{\mathrm{d}t}{t} - \frac{b_0}{s} + \frac{C^{-1}a_0}{s-k},$$

then changing $s \mapsto k-s$ turns $C \times (\dagger)$ into $(*)$.

□

Definition 8.13. Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(N,\chi) \subset M_k(\Gamma_1(N))$ be a modular form of level N. For a complex variable s, define its *Hecke L-function* by

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

which converges absolutely and uniformly in vertical strips in the region $\Re s > k$. If f is a cusp form, the series converges in the region $Res > \frac{k}{2} + 1$. Hence, $L(f,s)$ defines a holomorphic function in this region. (This follows from the estimate of the Fourier coefficients a_n, which is $O\left(n^{k-1+\varepsilon}\right)$ if f is an Eisenstein series, and $O\left(n^{\frac{k}{2}}\right)$ if f is a cusp form (Hecke's bound).)

Definition 8.14. For $f(\tau) \in M_k(N,\chi)$, define the *completed L-function* by

$$\Lambda(f,s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(f,s) \qquad \Re s > k.$$

Lemma 8.15. Recall that for $f \in M_k(N,\chi)$ and the Fricke element $\omega_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ we have

$$g(\tau) := f|_k \, \omega_N(\tau) = \left(\sqrt{N}\tau\right)^{-k} f\left(-\frac{1}{N\tau}\right) \in M_k(N,\overline{\chi}).$$

Set $F(t) := f\left(\frac{it}{\sqrt{N}}\right)$ and $G(t) := g\left(\frac{it}{\sqrt{N}}\right)$. Then

$$F\left(\frac{1}{t}\right) = i^k t^k G(t).$$

Theorem 8.16. Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_k(N,\chi)$. Then the completed L-function $\Lambda(f,s)$ is equal to the Mellin transform of f. More precisely,

$$\Lambda(f,s) = \int_0^{\infty} (F(t) - a_0)t^s \frac{\mathrm{d}t}{t},$$

and similarly for $\Lambda(g,s)$. Hence $\Lambda(f,s)$ and $\Lambda(g,s)$ converge absolutely and uniformly in compact subsets for $\Re s > k$ and define holomorphic functions in that region. Moreover, $\Lambda(f,s)$ and $\Lambda(g,s)$ have a meromorphic continuation to $\mathbb{C}$ such that

$$\Lambda(f,s) + \frac{a_0}{s} + \frac{i^k b_0}{k-s}, \qquad \Lambda(g,s) + \frac{b_0}{s} + \frac{(-i)^k a_0}{k-s}$$

are entire and bounded on vertical strips. In particular, if f is a cusp form, then $\Lambda(f,s)$ is entire. Finally, we have the functional equation

$$\Lambda(f,s) = i^k \Lambda(g, k-s).$$

Proof. The first statement is an easy calculation like the one for the Riemann ζ-function. The rest follows all from the Mellin principle applied to $F(t)$ and $G(t)$. □

Remark 8.17. The same statement holds for $f \in M_k(\Gamma_1(N))$.

Definition 8.18. Define

$$M_k^{\pm}(\Gamma_1(N)) := \left\{ f \in M_k(\Gamma_1(N)) : f|_k \, \omega_N = \pm(-i)^k f \right\}.$$

Then

$$M_k(\Gamma_1(N)) = M_k^+(\Gamma_1(N)) \oplus M_k^-(\Gamma_1(N)).$$

Note that $f \in M_k(N,\chi)$ can only be in the $\pm$-space if χ is trivial or quadratic. In this case, define $M_k^{\pm}(N,\chi)$ as before. Naturally, considering $f \in M_k(N,\chi)$, the form $f \pm i^k \, f|_k \, \omega_N \in M_k(N,\chi) \oplus M_k(N,\overline{\chi})$ gives elements in the $\pm$-space.

Theorem 8.19. Let $f \in M_k^{\pm}(\Gamma_1(N))$. Then

$$\Lambda(f,s) = \pm\Lambda(f, k-s).$$

Proof.

$$\Lambda(f,s) = i^k\Lambda(g,s) = i^k\Lambda(\pm(-i)^k f, s) = \pm\Lambda(f, k-s).$$

□

Remark 8.20. The importance of special values of the L-function $L(f,s)$ cannot be overstated, in particular, inside the critical strip $0<s<k$. The values at $s=1,2,...,k-1$ are known as the critical values (for which we expect/know significant arithmetical information). Note that for $f\in M_k^-(\Gamma_1(N))$, we immediately have $L\left(f,\frac{k}{2}\right)=0$ at the centre of the critical strip, that is, at the fixed point under $s\leftrightarrow k-s$. The generalised Riemann hypothesis states that all the non-trivial zeros of $L(f,s)$ are concentrated on the critical line $\Re s=\frac{k}{2}$.

We give another example of the principle discussed above without proof.

Definition 8.21 (Dirichlet L-series)**.** Let χ be a Dirichlet character mod N. Then

$$L(\chi,s):=\sum_{n=1}^{\infty}\chi(n)n^{-s}=\prod_p\frac{1}{1-\chi(p)p^{-s}}\qquad \Re s>1.$$

Complete by setting

$$L_\infty=\left(\frac{N}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{1+p}{2}\right)\qquad p\in\{0,1\}\text{ is defined by }\chi(-1)=(-1)^p.$$

Thus $\Lambda(\chi,s)=L_\infty(\chi,s)L(\chi,s)$.

Theorem 8.22. The completed Dirichlet L-series $\Lambda(\chi,s)$ is up to a constant the Mellin transform at $s=\frac{s+p}{2}$ of

$$\theta(\chi,\tau):=\sum_{n=-\infty}^{\infty}\chi(n)n^pe^{2\pi in^2\tau}.$$

One can then show similarly as above that for χ primitive, $\Lambda(\chi,s)$ has an analytic continuation to $\mathbb{C}$ (no poles) and satisfies the functional equation

$$\Lambda(\chi,s)=W(\chi)\Lambda(\overline{\chi},1-s),$$

where $W(\chi)=\frac{g(\chi)}{i^p\sqrt{N}}$ has absolute value 1, and $g(\chi)=\sum_{n=0}^{N-1}\chi(n)e^{\frac{2\pi in}{N}}$ is the Gauss sum.

9. Tuesday 2 December 2025

Definition 9.1. Define

$$\Delta_1^m(N)=\left\{\gamma\in\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{Z}):\gamma=\begin{pmatrix}1&*\\0&m\end{pmatrix}\bmod N,\ \det\gamma=m\right\}.$$

Note that for $\beta\in\Gamma_1(N)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&*\\0&1\end{pmatrix}\bmod N\right\}$ one has

$$\beta\Delta_1^m(N)=\Delta_1^m(N)=\Delta_1^m(N)\beta.$$

Definition 9.2 (Hecke operators)**.** Let $f\in M_k(\Gamma_1(N))$. Define

$$T_mf=m^{\frac{k}{2}-1}\sum_{\alpha_j\in\Gamma_1(N)\backslash\Delta_1^m(N)}f|_k\,\alpha_j,$$

where $\{\alpha_j\}$ is a system of right coset of $\Gamma_1(N)$ in $\Delta_1^m(N)$:

$$\Delta_1^m(N)=\bigcup_{\alpha_j}\Gamma_1(N)\alpha_j.$$

(Recall $f|_k\,\beta(\tau)=\det^{\frac{k}{2}}\beta j(\beta,\tau)^{-k}f(\beta\tau)$ for $\beta\in\mathrm{GL}_2^+(\mathbb{Q})$.)

Lemma 9.3.
1. The number of cosets is finite.
2. It is well defined, i.e., does not depend on the choice of α_j: for $\beta\in\Gamma_1(N)$ we have
$$f|_k\,\beta\alpha_j=(f|_k\,\beta)|_k\,\alpha_j=f|_k\,\alpha_j.$$
3. $T_mf\in M_k(\Gamma_1(N))$, i.e. $T_m:M_k(\Gamma_1(N))\to M_k(\Gamma_1(N))$. For $\beta\in\Gamma_1(N)$, then also $\{\alpha_j\beta\}$ is a set of representatives for $\Gamma_1(N)\backslash\Delta_1^m(N)$. Then
$$T_m\,f|_k\,\beta=m^{\frac{k}{2}-1}\sum_j(f|_k\,\alpha_j)|_k\,\beta=m^{\frac{k}{2}-1}\sum_j f|_k\,\alpha_j\beta=T_mf.$$
4. T_n maps cusp forms to cusp forms: $T_n:S_k(\Gamma_1(N))\to S_k(\Gamma_1(N))$.

Lemma 9.4.

$$\Delta_1^m(N) = \coprod_{\substack{d|m,\ (d,N)=1,\\ b \bmod d}} \Gamma_1(N)\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

here for each d, we set $a = \frac{m}{d}$ and $\sigma_a \in \Gamma_0(N)$ is any element in $\Gamma_0(N)$ with the right lower entry equal to a.

Proof. Verify that the terms on the right are indeed disjoint. Let $\alpha = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Delta_1^m(N)$. Choose g, h coprime with $ga' + hc' = 0$ and complete g, h to a matrix $\gamma = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \Gamma_0(N)$. Then $\det \gamma\alpha = m$ and $\gamma\alpha = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Replacing γ with $\pm T^j\gamma$, we can assume $\gamma\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with a, b, d as above. But now $\begin{pmatrix} 1 & * \\ 0 & m \end{pmatrix} = \alpha = \gamma^{-1}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ shows that the lower right entry of γ must be congruent to a. Hence $\gamma \in \Gamma_1(N)\sigma_d$. □

Remark 9.5. Let σ_d be any element in $\Gamma_0(N)$ as above. Recall $\Gamma_1(N)/\Gamma_0(N) = (\mathbb{Z}/N\mathbb{Z})^\times$, so

$$\Gamma_0(N) = \bigcup_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma_1(N)\sigma_d = \bigcup_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \sigma_d\Gamma_1(N),$$

so two such elements differ by an element in $\Gamma_1(N)$. Hence

$$f|_k\, \sigma_d \in M_k(\Gamma_1(N)) \qquad \text{for } f \in M_k(\Gamma_1(N)).$$

If $\sigma_d\gamma = \gamma'\sigma_d,\ \gamma, \gamma' \in \Gamma_1(N)$, then $f|_k\, \sigma_d\gamma = f|_k\, \gamma'\sigma_d = f|_k\, \sigma_d$.

Proposition 9.6. The Hecke operators T_m and $|_k\sigma_d$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ commute on $M_k(\Gamma_1(N))$:

$$(T_m f)|_k\, \sigma_d = T_m\left(f|_k\, \sigma_d\right).$$

Proof. We can take $\sigma_d = \Gamma_0(N)$ such that $\sigma_d = \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix} \bmod N$. Then one essentially computes

$$\sigma_d^{-1}\Delta_1^m(N)\sigma_d = \Delta_1^m(N).$$

Thus writing $\Delta_1^m(N) = \coprod_{\alpha_j} \Gamma_1(N)\alpha_j$, we see

$$\sigma_d \coprod_{\alpha_j} \Gamma_1(N)\alpha_j = \coprod_{\alpha_j} \Gamma_1(N)\alpha_j\sigma_d.$$

Hence

$$T_m\left(f|_k\, \sigma_d\right) = m^{\frac{k}{2}-1}\sum_{\alpha_j} f|_k\, \sigma_d\alpha_j = m^{\frac{k}{2}-1}\sum_{\alpha_j} f|_k\, \alpha_j\sigma_d = (T_m f)|_k\, \sigma_d.$$

□

Corollary 9.7. The Hecke operators for $\Gamma_1(N)$ preserve $M_k(N, \chi)$.

Proof. For $f \in M_k(N, \chi)$, one has $f|_k\, \sigma_d = \chi(d)f$. So $(T_m f)|_k\, \sigma_d = T_m\left(f|_k\, \sigma_d\right) = \chi(d)T_m f$. □

Definition 9.8. Let $f(q) = \sum_{n=0}^\infty a_n q^n \in \mathbb{C}[\![q]\!]$. Set

$$U_m f(q) = \sum_{m|n}^\infty a_n q^{\frac{n}{m}} = \sum_{n=0}^\infty a_{nm} q^n, \qquad V_m f(q) = \sum_{n=0}^\infty a_n q^{mn}.$$

We have for $q = e^{2\pi i\tau}$

$$U_m f(\tau) = \frac{1}{m}\sum_{j=0}^{m-1} f\left(\frac{\tau + j}{m}\right), \qquad V_m f(\tau) = f(m\tau).$$

Proposition 9.9. We have the following equality of operators on $M_k(N, \chi)$:

$$T_m = \sum_{d|m} \chi(d)d^{k-1}V_d \circ U_{\frac{m}{d}}.$$

Let $f = \sum_{n=0}^\infty a_n q^n \in M_k(N, \chi)$ and write $T_m f = \sum_{n=0}^\infty b_n q^n$. Then

$$b_n = \sum_{d|(m,n)} \chi(d)d^{k-1}a_{\frac{mn}{d^2}}.$$

In particular, for $m = p$ prime, we have
$$b_n = a_{pn} + \chi(p)p^{k-1}a_{\frac{n}{p}}.$$

Proof. By 9.3 we have
$$T_m f = n^{\frac{k}{2}-1}\sum f|_k\,\sigma_a\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}.$$
Since $f \in M_k(N,\chi)$ we first note $f|_k\,\sigma_a = \chi(a)f$. Now for each $a > 0$ and $d = \frac{m}{a}$, we have
$$\sum_{b=0}^{d-1} f|_k\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}(\tau) = \sum_{b=0}^{d-1} m^{\frac{k}{2}}d^{-k}f\left(\frac{a\tau+b}{d}\right) = m^{\frac{k}{2}}d^{-k+1}V_a\circ U_d f(\tau).$$
Hence
$$T_m f = m^{\frac{k}{2}-1}\sum_{a|m}\chi(a)d^{1-k}V_a\circ U_d,$$
where the sum is a priori only over a coprime to N. But the terms with $(a,N) > 1$ are 0 since $\chi(a) = 0$ in that case. □

Corollary 9.10. For $m \mid N$ we have $T_m = U_m$.

Proposition 9.11. The Hecke operators acting on $M_k(N,\chi)$ satisfy the following.
(1) $T_1 = 1$.
(2) If $(n,m) = 1$, then $T_{mn} = T_m T_n$. In particular T_n and T_m commute.
(3) If p is a prime dividing N, then $T_{p^\ell} = T_p^\ell$.
(4) If p is a prime not dividing N, then for $\ell \geq 2$,
$$T_{p^\ell} = T_{p^{\ell-1}}T_p - \chi(p)p^{k-1}T_{p^{\ell-2}} = T_pT_{p^{\ell-1}} - \chi(p)p^{k-1}T_{p^{\ell-2}}.$$

Corollary 9.12. All Hecke operators commute, that is, $T_mT_n = T_nT_m$ for all m,n.

Proof. This follows from the commutativity of T_n and T_m for $(n,m) = 1$ and the formula for T_{p^ℓ}. □

Proof of 9.11. We use the explicit formula
$$T_m = \sum_{d|m}\chi(d)\,d^{k-1}\,V_d\circ U_{m/d}.$$

We will need the following relations for U_m and V_n, which are immediate from the definition:
$$U_m\circ U_n = U_{mn}, \qquad V_m\circ V_n = V_{mn},$$
$$V_m\circ U_n = U_n\circ V_m \quad \text{if } (n,m)=1, \qquad U_m\circ V_m = 1, \qquad V_m\circ U_m \neq 1.$$
(1) $T_1 = 1$ is obvious since $\Delta_1^1(N) = \Gamma_1(N)$.
(2) follows easily from the commutativity of U_a and V_b for coprime a,b.
(3) is clear: $T_{p^\ell} = U_{p^\ell} = U_p^\ell = T_p^\ell$.
(4) First we show $T_{p^\ell} = T_{p^{\ell-1}}\,T_p - \chi(p)\,p^{k-1}\,T_{p^{\ell-2}}$. We directly see
$$T_{p^\ell} = T_{p^{\ell-1}}\,U_p + \chi(p^\ell)p^{\ell(k-1)}V_{p^\ell} = T_{p^{\ell-1}}\,T_p - T_{p^{\ell-1}}\chi(p)p^{k-1}V_p + \chi(p^\ell)p^{\ell(k-1)}V_{p^\ell}.$$
The second and third term give
$$-\left[T_{p^{\ell-2}}\,U_p + \chi(p^{\ell-1})p^{(\ell-1)(k-1)}V_{p^\ell}\right]\chi(p)p^{k-1} + \chi(p^\ell)p^{\ell(k-1)}V_{p^\ell} = -\chi(p)p^{k-1}T_{p^{\ell-2}},$$
since $U_pV_p = 1$. $T_{p^\ell}T_pT_{p^{\ell-1}} - \chi(p)p^{k-1}T_{p^{\ell-2}}$ goes similarly (verify).

□

We now make the following observation that $M_k(\Gamma_0(N)) = M_k(N,\chi_0)$, where χ_0 is the trivial character mod N, and always view $M_k(\Gamma_0(N))$ in this way.

Lemma 9.13. We have the following formal power series identity when operating on $M_k(N,\chi)$:
$$\sum_{\ell=0}^{\infty} T_{p^\ell}X^\ell = \frac{1}{1 - T_pX + \chi(p)p^{k-1}X^2},$$
that is,
$$\left(1 - T_pX + \chi(p)p^{k-1}X^2\right)\left(\sum_{\ell=0}^{\infty} T_{p^\ell}X^\ell\right) = 1.$$
In fact, it is equivalent to the formula T_{p^ℓ} given in 9.11.

Proof. When multiplying out $\left(1 - T_p X + \chi(p)p^{k-1}X^2\right)\left(\sum_{\ell=0}^{\infty} T_{p^\ell} X^\ell\right)$ and equating coefficients, we see that the claim is equivalent to $T_1 = 1$ and $T_{p^\ell} = T_{p^{\ell-1}} T_p - \chi(p)p^{k-1}T_{p^{\ell-2}}$, which are 9.11(1) and (4). Note here $\chi(p) = 0$ if $p \mid N$, so we do not have to distinguish the cases $p \mid N$ and $p \nmid N$. □

Proposition 9.14. The generating series of the Hecke operators satisfies the following formal Euler product:

$$\begin{aligned}\sum_{n=1}^{\infty} T_n n^{-s} &= \prod_p \frac{1}{1 - T_p p^{-s} + \chi(p)p^{k-1-2s}} \\ &= \prod_{p\nmid N} \frac{1}{1 - T_p p^{-s} + \chi(p)p^{k-1-2s}} \prod_{p\mid N} \frac{1}{T_p p^{-s}}.\end{aligned}$$

In fact, this is equivalent to the relations given in 9.11.

Proof. For $n = p_1^{a_1} \cdots p_k^{a_k}$, we have $T_n n^{-s} = T_{p_1^{a_1}} p_1^{-a_1 s} \cdots T_{p_k^{a_k}} p_k^{-a_k s}$ by 9.11. But now

$$\sum_{a_1=0}^{\infty} T_{p_1}^{a_1} \left(p_1^{-s}\right)^{a_1} = \frac{1}{1 - T_p p^{-s} + \chi(p)p^{k-1-2s}}$$

by the previous lemma. □

Jiewei Xiong, Department of Mathematics and Statistics, Mathematics Building, University of Reading, Whiteknights campus, Reading RG6 6AX, United Kingdom
Email address: jiewei.xiong@pgr.reading.ac.uk