MAXIMAL SPECTRA AND ULTRASCHEMES

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Abstract. The title refers to [EGA IV:3, §10.9]. In this note we explain why when working with schemes of finite type over a field, it's "natural" to ignore nonclosed points and replace the Spec's in our language with Spm's.

We first recall some basic concepts in our modern common language of schemes. All rings are commutative with identity, and in particular Ring denote the category of commutative rings with identity.

Definition 1. For a ring A, the *spectrum* of A, denoted by $\operatorname{Spec} A$, is the set of all prime ideals of A.

For $S \subset A$, define $V(S) := \{\mathfrak{p} \in \operatorname{Spec} A : S \subset \mathfrak{p}\}$, and for $f \in A$, define $D(f) := \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$.

Endow $\operatorname{Spec} A$ with the *Zariski topology*: a subset of $\operatorname{Spec} A$ is closed if it equals $V(S)$ for some $S \subset A$.

Definition 2. $x \in \operatorname{Spec} A$ is a *closed point* if $\{x\}$ is a closed set.

Note that by definition of the Zariski topology, $\{x\}$ is closed if and only if $\{x\} = \{\mathfrak{p} \in \operatorname{Spec} A : S \subset \mathfrak{p}\}$ for some $S \subset A$, that is, the only prime ideal of A that includes S is x. This is equivalent to that x is a maximal ideal of A.

The *maximal spectrum* of A, denoted by $\operatorname{Spm} A$, is the set of all maximal ideals of A, which then corresponds exactly to the closed points of $\operatorname{Spec} A$.

Definition 3. A topological space X together with a (structure) sheaf of rings $\mathcal{O}_X$ on it is called a *ringed space*. For now we can ignore the definition of a *sheaf of rings* and simply think of them as a function that takes the open sets of X as inputs and has rings as outputs.

A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is defined naturally: it's

(1) a continuous map of topological spaces $\pi : X \to Y$ (the morphisms of Top), together with
(2) a natural transformation of functors $f : \mathcal{O}_Y \to \pi^* \mathcal{O}_X$ (the morphisms of $\mathsf{Ring}^{\mathcal{D}}$, the category of functors $\mathcal{D} \to \mathsf{Ring}$, where $\mathcal{D}$ is the poset category of open sets of Y).

Definition 4. Define the structure sheaf $\mathcal{O}_{\operatorname{Spec} A} : \mathcal{C} \to \mathsf{Ring}$ on the topological space $\operatorname{Spec} A$ (where $\mathcal{C}$ is the poset category of open sets of $\operatorname{Spec} A$) as follows: note that $D(f)$ forms a basis of the Zariski topology, and let $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = S^{-1}A$ where

$$S = \{g \in A : D(f) \subset D(g)\}.$$

This sheaf together with the topological space $\operatorname{Spec} A$ is an *affine scheme*, denoted by $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, or simply $\operatorname{Spec} A$ by abuse of notation.

Recall that there is an isomorphism $\mathcal{O}_{\operatorname{Spec} A}(D(f)) \xrightarrow{\sim} A_f$ where A_f means A localised at $\{1, f, f^2, \ldots\}$.

It's tempting to expect *a morphism of affine schemes* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ to be simply a morphism of them as ringed spaces. However, we also expect these morphisms to behave well with the underlying rings A and B; that is, since each ring homomorphism $B \to A$ gives rise to a Zariski continuous map $\operatorname{Spec} A \to \operatorname{Spec} B$ (preimage of a prime ideal under a ring homomorphism is prime), each morphism of affine schemes should arise from some ring homomorphism as well.

More precisely, let $\phi : B \to A$ be a ring homomorphism, and let $(\pi, f) : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of ringed spaces where π is induced by ϕ, and for $g \in B$,

$$f_{D(g)} : \mathcal{O}_{\operatorname{Spec} B}\left(D(g)\right) \to \mathcal{O}_{\operatorname{Spec} A}\left(\pi^{-1}\left(D(g)\right)\right)$$

is identified with the ring homomorphism $B_g \to A_{\phi(g)}$, again induced by ϕ. This (π, f) is denoted by $\left({}^a\phi, \widetilde{\phi}\right)$ in [EGA I, 1.6.1], which is the form we expect every morphism of affine schemes to be of. It turns our we do need to add a condition to our definition, and first we consider an example of a morphism (π, f) that doesn't satisfy this condition and therefore is not of the form $\left({}^a\phi, \widetilde{\phi}\right)$.

Example 5. Let k be a field and consider a morphism of ringed spaces $(\pi, f) : \operatorname{Spec} k(x) \to \operatorname{Spec} k[y]_{(y)}$. Since $k(x)$ is a field, the only prime ideal is (0), and any prime ideal of $k[y]_{(y)}$ correspond to a prime ideal $\mathfrak{p}$ of $k[y]$ with $\mathfrak{p} \cap k[y] \backslash (y) = \varnothing$, i.e. $\mathfrak{p} \subset (y)$, but there are only two such $\mathfrak{p}$'s: (0) and (y). We then write $\operatorname{Spec} k[y]_{(y)} = \{[(0)], [(y)]\}$, where $[(y)]$ is the unique maximal ideal of the local ring $k[y]_{(y)}$. Note that $[(0)]$ is not closed: it is impossible that for some set $S \subset k[y]_{(y)}$ we have that the only prime ideal including S is the zero ideal. So for π to be continuous, $\pi((0))$ can only be $[(y)]$. Define f globally by sending k identically to k and y to x.

We claim this does not arise from any ring homomorphism $\phi : k[y]_{(y)} \to k(x)$. Indeed, suppose it does arise from some ϕ. By our definition of π, ϕ sends y to 0. Now let $g \in k[y]_{(y)}$.

(1) If $g \in [(y)]$, then $k(x)_{\phi(g)} = k(x)_0 = 0$, contradicting our definition of f.
(2) If $g \notin [(y)]$ then g is a unit, so ${k[y]_{(y)}}_g = k[y]_{(y)}$ and $k(x)_{\phi(g)} = k(x)$. Hence ϕ is, by our definition of f, the homomorphism that fixes k and sends y to x, contradicting our definition of π.

The problem with the map above is that f does not send the unique maximal ideal of $k[y]_{(y)}$, i.e. $[(y)]$, into the unique maximal ideal of $k(x)$, i.e. (0).

EGA provided a condition we impose on (π, f) to avoid the situation above.

Theorem 6 ([EGA I, Théorème 1.7.3])**.** A necessary and sufficient condition that (π, f) is of the form $\left({}^a\phi, \widetilde{\phi}\right)$ is that for each $\mathfrak{p} \in \operatorname{Spec} A$, the map of *stalks* $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{\operatorname{Spec} B, f(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} : B_{f(\mathfrak{p})} \to A_{\mathfrak{p}}$ induced is a *local ring homomorphism*, that is, ${f_{\mathfrak{p}}^{\#}}^{-1}(\mathfrak{p}) = f(\mathfrak{p})$.

This is the condition we commonly use:

Definition 7. A *morphism of affine schemes* $\operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism between them as ringed spaces such that the condition of 6 is satisfied.

The categorical result is that the two functors

$$A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

and

$$(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \mapsto \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$$

give an equivalence between Ring and $\mathsf{Aff}^{\mathrm{op}}$, the opposite category of affine schemes ([EGA I, Corollaire 1.7.4]). The punchline is: the theory of affine schemes is exactly the theory of commutative rings!

We are now interested in affine schemes of finite type over a field:

Definition 8. A ring homomorphism $\phi : A \to B$ is *of finite type* if it makes B a finitely generated A-algebra, that is, there is a surjective homomorphism $A[x_0, \dots, x_n] \to B$ for some $n \in \mathbb{N}$ that factors through ϕ.

An affine scheme $\operatorname{Spec} A$ is *of finite type* over another affine scheme $\operatorname{Spec} B$ if there is a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ that arises from a ring homomorphism $B \to A$ of finite type.

$\operatorname{Spec} A$ is *of finite type* over a field k if it's of finite type over $\operatorname{Spec} k$, or, equivalently, if A is a finitely generated k-algebra.

They are sometimes also called *algebraic* affine k-schemes, and specifically when we say *algebraic groups* in a scheme-theoretic context, we mean *algebraic group k-schemes*.[Mil22, p. 4] (Group schemes are schemes whose *functor of points* takes values in Grp, whatever this means.)

In [Mil22, p. 2], the author says

> as we always work with schemes of finite type over a base field k, it is natural to ignore the nonclosed points (which we do),

that is, they only consider $\operatorname{Spm} A$ instead of $\operatorname{Spec} A$. In [Per08, B.1.a], also dealing with affine schemes $\operatorname{Spec} A$ of finite type over k, the author even says they sometimes replace the notation $\operatorname{Spm} A$ altogether with $\operatorname{Spec} A$, humorously adding "if we are sure Grothendieck is not listening". The key question is:

Question 9. If we identify a ring A with an affine scheme $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, what's special about a finitely generated k-algebra A that enables us to identify it with only $(\operatorname{Spm} A, \mathcal{O}_{\operatorname{Spm} A})$? (And what even is $\mathcal{O}_{\operatorname{Spm} A}$?)

In fact, Grothendieck himself provided a justification for this choice in [EGA IV:3, §10.9], titled "maximal spectra and ultraschemes" (we customarily translate "préschéma" to "scheme"), although he did avoid to use the results of the section later [les résultats de ce numéro ne seront pas utilisés par la suite]. There they develop in the full generality of *jacobson schemes*. For us we first need to know:

Definition 10. A ring A is *jacobson* if every radical ideal of A is the intersection of the maximal ideals containing it.

Lemma 11 ([Stacks, 00G2])**.** If every prime ideal of A is the intersection of the maximal ideals containing it, then A is jacobson.

Proof. Recall that every radical ideal is the intersection of the prime ideals containing it.[AM69, 1.14] □

Definition 12 ([Stacks, 01P3]). An affine scheme Spec A is *jacobson* if A is jacobson.

Lemma 13 ([Stacks, 02J6]). Any affine scheme of finite type over a field is jacobson.

Proof. Let A be a finitely generated k-algebra and write $A = k[x_0, \dots, x_n]/J$ for some $n \in \mathbb{N}$ and $J \lhd k[x_0, \dots, x_n] =: R$. We need to prove A is a jacobson ring.

We first prove that R is jacobson. Let $I \lhd R$ be radical. The inclusion $I \subset \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$ is trivial, and to prove $I \supset \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$, we show the contrapositive: $\forall f \in R,\ f \notin I \implies f \notin \bigcap_{\mathfrak{m} \supset I} \mathfrak{m}$. So let $f \in R \backslash I$ and we want to find a maximal ideal $\mathfrak{m} \lhd R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Consider the localisation $(R/I)_f$. This ring is nonzero, since otherwise $f^k \in I$ for some $k \in \mathbb{N}$ and since J is radical, $f \in I$, a contradiction. Hence choose a maximal ideal $\mathfrak{m}'$ of $(R/I)_f$, and we claim the preimage $\mathfrak{m}$ of $\mathfrak{m}'$ in R is the $\mathfrak{m}$ we want to find. By [AM69, 3.11(iv)], $f \notin \mathfrak{m}$, and $I \subset \mathfrak{m}$ by construction. It remains to see $\mathfrak{m}$ is indeed maximal, but this is clear: we have $k \subset R/\mathfrak{m} \subset (R/I)_f/\mathfrak{m}'$, where $(R/I)_f/\mathfrak{m}'$ is a finite extension of k, so $R/\mathfrak{m}$ is forced to be a field.

It's then clear that A is jacobson by the correspondence of ideals between $A = R/J$ and R. □

Lemma 14. Let A, B be a jacobson rings with $\phi : A \to B$ of finite type. If $\mathfrak{m}$ is a maximal ideal of B, then $\phi^{-1}(\mathfrak{m}) \lhd A$ is maximal.

Proof. Write $B = A[x_0, \dots, x_n]/I$ and consider ϕ to be the inclusion. Then $\phi^{-1}(\mathfrak{m}) = \mathfrak{m} \cap A =: \mathfrak{m}'$. The rings $B/\mathfrak{m}$ and $A/\mathfrak{m}'$ are jacobson again by correspondence, and $B/\mathfrak{m}$ is a finitely generated $A/\mathfrak{m}'$-algebra via the map induced by ϕ. Now $A/\mathfrak{m}'$ is clearly a domain, and we can choose an $f \neq 0$ such that the localisation $(A/\mathfrak{m}')_f$ is a field by [Stacks, 00FY]. But then (0) is the only maximal ideal of $(A/\mathfrak{m}')_f$, so it's the only maximal ideal of $A/\mathfrak{m}'$ as well by [Stacks, 00G6]. □

Now consider the subcategory $\mathcal{J} \hookrightarrow \mathsf{Aff}$ of jacobson affine schemes with morphisms Spec $A \to$ Spec B being morphisms of affine schemes of finite type, that is, the corresponding ring homomorphism (via 6) $B \to A$ is of finite type. Consider another category $\mathcal{U}$ of *ultra affine schemes*; the objects are ringed spaces $(\operatorname{Spm} A, \mathcal{O}_{\operatorname{Spm} A})$ where A is jacobson and

(1) the topology of Spm A is induced as a subspace of Spec A; in particular the
$$D^m(f) := D(f) \cap \operatorname{Spm} A = \{\mathfrak{m} \in \operatorname{Spm} A : f \notin \mathfrak{m}\}$$
form a basis of this topology, and

(2) $\mathcal{O}_{\operatorname{Spm} A}$ is defined in a similar way as $\mathcal{O}_{\operatorname{Spec} A}$, that is, $\mathcal{O}_{\operatorname{Spm} A}(D^m(f)) = S^{-1}A$ where
$$S = \{g \in A : D^m(f) \subset D^m(g)\},$$

and the morphisms are again those of finite type (morphisms of ultra affine schemes are defined in an analogous way as morphisms of affine schemes; in particular, due to 14, we can similarly require them to be of the form $\left({}^b\phi, \widehat{\phi}\right)$, induced by a ring homomorphism ϕ).

Then

Theorem 15 ([EGA IV:3, Proposition 10.9.6]). There is an equivalence of categories between $\mathcal{J}$ and $\mathcal{U}$.

Proof. Clearly $\mathcal{J}$ is equivalent to the opposite category $\mathcal{J}'^{\mathrm{op}}$ of jacobson rings with morphisms being ring homomorphisms of finite type. We then show that the functor $F : \mathcal{J}'^{\mathrm{op}} \to \mathcal{U}$ that takes a jacobson ring A to $(\operatorname{Spm} A, \mathcal{O}_{\operatorname{Spm} A})$ and takes a ring homomorphism ϕ of finite type to $\left({}^b\phi, \widehat{\phi}\right)$ gives an equivalence of categories.

First, we show that F is fully faithful, that is, the map between Hom sets
$$\operatorname{Hom}_{\mathcal{J}'}(B, A) \to \operatorname{Hom}_{\mathcal{U}}(\operatorname{Spm} A, \operatorname{Spm} B)$$
induced by F is bijective.

(1) It's injective: we need to show that if ϕ, ψ both induce $\left({}^b\phi, \widehat{\phi}\right)$, then $\phi = \psi$. By definition, the data of $\left({}^b\phi, \widehat{\phi}\right)$ gives a ring map
$$\widehat{\phi}_{D(1)} : \mathcal{O}_{\operatorname{Spm} B}(\operatorname{Spm} B) = B \to A = \mathcal{O}_{\operatorname{Spm} A}\left(\left({}^b\phi\right)^{-1}(\operatorname{Spm} B)\right),$$
which uniquely gives the ϕ that induces $\left({}^b\phi, \widehat{\phi}\right)$.

(2) It's surjective: we need to show every $\left({}^b\phi, \widehat{\phi}\right)$ is induced by some ϕ, which is a tautology.

It remains to show that F is essentially surjective, which follows again by construction. □

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