Jean Dieudonné

For the honour of the human mind

Mathematics
today

Translator's note

The following is a translation and LaTeXitification of

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For bibliography, we let BibLaTeX count so the numbering is slightly different from original. We also add bibliographic information in the references to help the reader navigate to the source.

We claim no originality except errors and inaccuracies in translation (mostly done with ChatGPT).

February 2026 Jiewei Xiong

... Mr. Fourier believed that the main purpose of mathematics was public utility and the explanation of natural phenomena; but a philosopher like him should have known that the sole purpose of science is the honour of the human mind, and that under this heading, a question of numbers is as valid as a question of the world system.

C. G. J. Jacobi, letter (in French)
to Legendre, 2 July 1830.
Gesammelte Werke, Vol. I, Berlin: Reimer, 1881, p. 454.

To Odette and Françoise

Contents

Introduction

This book is intended exclusively for readers who are interested in science for various reasons, but *who are not professional mathematicians*. Experience shows that almost invariably, while they read or listen with pleasure to presentations on the natural sciences and feel that they gain from them information that enriches their view of the world, an article on contemporary mathematics seems to them to be written in an incomprehensible jargon and to deal with notions that are too abstract to be of the slightest interest.

The purpose of this book is to try to explain the reasons for this lack of understanding and perhaps to dispel it. Since it is not possible to deal with mathematics without assuming a minimum of prior knowledge of the subject, I must address readers who have followed a mathematical education at the level of a scientific baccalaureate. I have included a few supplements in the form of appendices to the chapters, intended for those who have pursued their scientific studies a little further (at least two years of university); but understanding the text proper does not require reading any of these supplements, which can therefore be skipped without inconvenience. What is required is only a small effort of sustained attention in following the chain of reasoning, each link of which, taken in isolation, is entirely "elementary".

What I propose to show is that the very nature of this "minimum baggage" lies at the source of the difficulties in understanding contemporary mathematics. *Nothing that is taught in mathematics at the secondary-school level was discovered after 1800*[1]. With very few exceptions, the same is true of the mathematical knowledge deemed necessary for future scientists in all fields of the natural sciences, with the exception of physics. But even for physicists other than those working in quantum theories or relativity, I believe that an experimentalist uses scarcely more mathematics than Maxwell knew in 1860.

Everyone knows that the natural sciences have progressed in an extraordinary way since the beginning of the nineteenth century, and that the "cultivated" public has been able to follow this dizzying ascent—perhaps from a certain distance, but without losing its footing—thanks to intelligent simplifications that preserved the essential new ideas. Mathematics has progressed no less, but, outside the community of mathematicians, almost no one has noticed it. The reason is that, on the one hand, as was said above, most sciences have little need, even today, for anything beyond "classical" mathematics; and on the other hand, a genuine mutation has taken place within mathematics: the creation of new "mathematical objects", entirely different from the "classical" objects of numbers and "figures". Their much more abrupt abstraction (since it no longer rests on sensible "images") has turned away those who saw no use for them.

What I want to convince the reader of good will of is that this increased

[1] But they do not teach everything that was discovered before that date, far from it!

abstraction does not at all originate in some perverse desire on the part of mathematicians to isolate themselves without reason from the scientific community by means of a hermetic language. They had to solve problems inherited from the "classical" age, or arising directly from the new acquisitions of physics. They discovered that this could be done, but only on the condition of creating new objects and new methods, whose abstract character was indispensable to their success.

I shall therefore attempt to show that it was under the constraint of the deep nature (often hidden until then) of classical objects and relations that mathematicians, between 1800 and 1930, forged the new "abstract" tools that made it possible for them—and for their present-day successors—to solve problems that had seemed beyond reach. Contemporary mathematicians have known how to make imaginative use of these tools and to develop their application, but they did not create them, contrary to what is sometimes believed.

I believe that it is not possible to understand today's mathematics unless one has at least a rough idea of its history. After two chapters outlining the situation of mathematics and mathematicians in the contemporary world, as well as the relations between mathematics and the other sciences, Chapter III offers a rapid survey of classical mathematics, from Euclid to Gauss, emphasizing the great original idea of the Greeks: the character of "objects of thought" that they indelibly attributed to mathematical notions. Chapter IV gives a few examples—among a multitude of others—of problems that arose in classical mathematics and that gave rise to important research in the nineteenth century.

Chapter V is the central point of my argument. Once again, through a few examples accessible to the reader, one sees how, in the nineteenth century, by analysing the problems inherited from the previous century, their true nature gradually emerged, leading to their complete or partial solution. But the price to be paid is the necessity of moving away from the half-"concrete" character of classical mathematical objects; one must understand that, for these objects, what is essential lies not in their apparent particularities, but in the relations they bear to one another. These relations are often the same for objects that appear very different, and they must therefore be expressed in a way that takes no account of such appearances. If one wishes, for example, to state a relation that can be defined equally well between numbers and between functions, this can only be done by introducing objects that are neither numbers nor functions, but that can be specialised at will into numbers, or functions, or many other kinds of mathematical objects. It is these "abstract" objects that are studied in what has come to be called mathematical structures, the simplest of which are described in Chapter V.

Finally, Chapter VI relates the difficulties that arose in the nineteenth century in formulating definitions and theorems with complete precision, leaving no room for doubt. To achieve this, Euclid's *Elements* served as a model in all branches of mathematics, once their shortcomings had been corrected and the purity of the Greeks' original conceptions—too long neglected in the euphoria of the incredible successes of the applications of mathematics to the experimental sciences—had been recovered.

Readers fond of sensational books will be disappointed by this one: they will find in it no explosive theses bristling with ingenious paradoxes. I have confined myself to presenting facts concerning mathematics, not opinions, and I have refrained from entering into polemics; I have also carefully avoided attempting any prediction about the future evolution of the notions I discuss.

Bibliography with commentary

[1] G. Choquet. 'La naissance de la théorie des capacités : réflexion sur une expérience personnelle'. In: *La Vie des Sciences – Revue de l'Académie des Sciences* 3 (1986), pp. 385–397. Zbl: 0607.01017.

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[3] J. Dieudonné et al. *Abrégé d'histoire des mathématiques 1700–1900*. Ed. by J. Dieudonné. Nouvelle édition. Paris: Hermann, 1986. Zbl: 0656.01001.

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[13] H. Poincaré. *Dernières pensées*. Paris: Flammarion, 1920. Zbl: 0121.00102.

[14] A. Weil. *Œuvres scientifiques, in 3 volumes*. Un Savant, une époque. New York–Heidelberg–Berlin: Springer, 1985. Zbl: 0424.01027.

Since I am not addressing scholars, I did not think it necessary to burden the text with a multitude of references, and I have limited the Bibliography to a few works that are in keeping with the spirit of this book. I hope readers will take me at my word when I say that I could cite a source for each of my assertions concerning the history of mathematics.

One cannot encourage readers too strongly to acquaint themselves with this history in greater detail. I would point in particular to book [2], enhanced with numerous illustrations and requiring no more advanced knowledge than that of the baccalaureate. At a somewhat higher level (two or three years of university), I allow myself to recommend volume [3] for the history after 1700.

Reading Diophantus [4] and Euclid [6] is always instructive; the English translation by T. Heath [5] is accompanied by very extensive notes and explanations of a mathematical or philological nature.

There are not many mathematicians who have published reflections on their science nourished by their personal experience. The enormous correspondence of Euler [7] allows us to know his life and his working methods better, and one always reads with interest the many articles by Poincaré on science, collected in four volumes [10, 11, 12, 13]. In a generation closer to our own, G. H. Hardy, at the end of his life, wrote a very engaging *Apology* [8], in which the man constantly shows through the mathematician.

One of the greatest mathematicians of my generation, A. Weil, had the fortunate idea of accompanying the edition of his *Œuvres* [14] with fairly extensive commentaries on each of his books or articles; this constitutes a fascinating "logbook" that describes, almost day by day, the evolution of the thought of a creative mathematician. Even though a particularly broad mathematical culture is required to appreciate it fully, one cannot fail to profit from these reminiscences, written without braggadocio or false modesty.

Quite similar in spirit and equally enriching is the account recently given by G. Choquet [1] of the path that led him to one of his fine discoveries, that of generalised capacities.

Finally, I would like to draw attention to a very recent heroic attempt by the Franco-American mathematician S. Lang to convey to the heterogeneous public of the Palais de la Découverte what "the beauty of mathematics" is, in three lectures followed by discussions [9]. The lecturer's good will—he is a "pure" mathematician if ever there was one—in trying to adapt himself to his audience, and his composure in the face of their unpredictable reactions, make it a performance that is as lively and entertaining as it is instructive.

Chapter I

Mathematics and mathematicians

1. The conception of mathematics

The position of mathematics among human activities is paradoxical. In our time, almost all inhabitants of the "developed" countries know that it is an important discipline, necessary in most areas of science and technology, and that only a good knowledge of it makes it possible to practice an ever-increasing number of professions. But if one asks, "What is mathematics?" or "What does a mathematician do?", it is extremely rare to obtain from the person questioned anything other than an absurd answer, unless they have received mathematical instruction at least up to the end of the first two years of university. Even eminent figures in other sciences often have only utterly distorted notions of the work of mathematicians.

Since everyone's first contact with mathematics is numerical calculation in primary school, the most widespread idea is that a mathematician is someone who is a virtuoso of such calculations. With the advent of computers and their languages, one is now inclined to think that a mathematician is someone who is particularly skilled at "programming" them and devotes all of his time to it. Engineers, always in search of optimal values for the quantities that interest them, see mathematicians as the custodians of a treasure trove of formulas that they are supposed to supply on demand. But the most fallacious of the common conceptions is that, while being overwhelmed by the advances that the media endlessly describe to them in all the other sciences, almost all our contemporaries remain convinced that there is nothing left to discover in mathematics, and that the mathematician merely confines himself to teaching the legacy of past centuries.

Should one stop there, and confine oneself to noting, with A. Weil ([14, II, p. 465]), that "mathematics has this peculiarity of not being understood by non-mathematicians"? I believe that one must at least try to find the reasons for this lack of understanding. There do in fact exist numerous periodicals devoted to the popularization of recent scientific discoveries, addressed to a wide readership and to all levels of education. With a few exceptions, to which we shall return (Chap. VI, 5, C), one finds in them nothing about recent advances in mathematics (which may lead one to believe that there have been none!); on the other hand, they deal at length with astrophysics, geology, chemistry, molecular biology, and even atomic or nuclear physics. Apparently, with a few drawings in which details are eliminated and the essential objects clearly highlighted, together with a few explanations that simplify to the extreme what complex and delicate experiments have revealed to specialists, one succeeds in giving the readers of these publications the illusion that they understand what an atom, a gene, or a galaxy is—even though the scale of the magnitudes involved, ranging from the ångström (10^{-10} metre) to the light-year ($3 \cdot 10^{15}$ metres), is inaccessible to our imagination.

But let us take one of the most fruitful theories of today's mathematics, what is called "sheaf cohomology". Invented in 1946, it is roughly contemporaneous with the "double helix" of molecular biology, and has made possible advances of comparable scope. Yet I would be absolutely incapable of explaining what it consists of to someone who had not followed at least the mathematics courses of the first two years of university; and even for a well-gifted student at that level, the explanations would take several hours, and the way the theory is actually used would take much longer still. The reason is that there are no longer any explanatory drawings here: before arriving at the theory in question, one must have assimilated a dozen other notions just as abstract—topologies, rings, modules, homomorphisms, and so on—none of which can be made "visual".

The same remarks can be made for almost all the notions that underlie the major mathematical theories of today; a list of the latter will be found in Chapter V, 5, A. I must therefore resign myself to the following: if I wish to speak of what is accessible at the level of the baccalaureate, I can allude only to the elementary concepts in three of these headings (out of twenty-one): algebra, number theory, and set theory, with from time to time a supplement intended for readers who have some familiarity with the infinitesimal calculus. Starting from these notions, I shall attempt to explain how they *necessarily* evolved and brought to light other underlying notions that are far more abstract, thereby acquiring an incomparably greater effectiveness in solving mathematical problems. To go further would be to line up strings of words in which the non-mathematical reader would very quickly lose his footing; the list I mentioned is there only to give an idea of how *infinitesimal* the number of theorems I shall discuss is, compared with the immensity of present-day mathematical knowledge.

2. The life of mathematicians

The term "musician", in everyday usage, may designate a composer, a performer, or a teacher of music, with possible overlaps among these activities. Similarly, by "mathematician" one may mean a teacher of mathematics, a user of mathematics, or a creative mathematician; and the common belief is that this last species is now extinct. Since my aim is to make the genesis and the nature of today's mathematics understood (most of which, as we shall see [Chap. V, 5, A], goes back no further than about 1840), it is nevertheless almost exclusively this last category that will be discussed here. A mathematician, in what follows, will therefore be defined as someone *who has published at least the proof of one nontrivial theorem* (the meaning of the term "trivial" will be specified later [§4]).

It seems to me that it is not without interest first to invite the reader on a short journey into the unknown country of mathematicians, before attempting to explain what they do.

There is every reason to think that the aptitude for mathematical creation is independent of race and does not appear to be transmitted to any great extent by heredity: mathematicians who are the children of mathematicians have always been a rare exception. But in the emergence of mathematical talent, the surrounding social environment plays an important role. There is no example, even among the greatest geniuses, of a mathematician rediscovering *ab ovo* the mathematical knowledge of his time, and what has been told in this regard about

Pascal is nothing more than a legend. The social environment must therefore be such that the future mathematician has been able to receive at least an elementary education that brings him into contact with genuine proofs, awakens his curiosity, and then allows him to familiarise himself with the mathematics of his time, provided that he has access to the works in which it is presented. Until the end of the eighteenth century, there was no truly organised higher education in mathematics, and from Descartes and Fermat to Gauss and Dirichlet, the great mathematicians almost all trained themselves without a master, through the reading of the works of their illustrious predecessors (which remains an excellent exercise!). Even today, it is plausible that many mathematical talents never manage to manifest themselves for lack of a favourable social atmosphere, and there is nothing surprising in the fact that no mathematicians are known in societies that are little "developed". And in more advanced countries, the type of elementary education may fail to be conducive to the emergence of mathematical vocations when it is subject to religious or political constraints, or to exclusively utilitarian concerns, as was the case in the United States until well into the twentieth century. Secondary education there was indeed in the hands of local authorities, who often judged it more important for adolescents to learn how to drive a car or type than to learn Latin or Euclidean geometry. I have known American mathematicians, who later became famous, who, having been born in small towns, had never seen a mathematical proof until the age of eighteen; drawn to science, they had turned at university toward an engineering career, and it was by chance that they were there confronted with authentic mathematical instruction and thus discovered their true vocation.

In societies where education can be provided to fairly broad segments of the population (if necessary with the help of scholarships), the social origins of mathematicians are very diverse. They may come from the nobility, like Fagnano, Riccati, and d'Alembert; from the upper bourgeoisie, like Descartes, Fermat, Pascal, Kronecker, Jordan, Poincaré, and von Neumann; or, on the contrary, from very humble backgrounds, like Roberval, Gauss, Élie Cartan, and Lebesgue. Most are born into middle-class families, often rather hard-working ones.

The mathematical vocation most frequently awakens around the age of fifteen, but it can be delayed by an education that leaves no place for the idea of proof, as was the case in the United States mentioned above. Nevertheless, contrary to a fairly widespread opinion, the beginning of the creative period rarely occurs before the age of 23 to 25; the cases of Pascal, Clairaut, Lagrange, Gauss, Galois, Minkowski, and in our own day Deligne—who owe notable mathematical discoveries to work done before the age of twenty—are exceptional. As for mathematical longevity, another common opinion is that, as Hardy says [8, p. 14], "mathematics is a young man's game". It is indeed true that many major discoveries have been made by mathematicians whose age scarcely exceeded thirty. But for many of the greatest figures, such as Poincaré, Hilbert, or H. Weyl, the creative period extends, just as fruitfully, up to around fifty or fifty-five; others, such as Killing, Emmy Noether, Hardy himself (as he acknowledges [8, p. 58], or more recently Zariski and Chevalley, give the best of themselves around the age of forty. Finally, a few mathematicians especially favoured by the gods, such as Kronecker, Élie Cartan, Siegel, A. Weil, J. Leray, and I. Gelfand, have continued to prove beautiful theorems after the age of sixty. But just as in many sports a champion can hardly hope to remain one after thirty, most mathematicians must resign themselves to seeing their creative imagination dry up after fifty-five or sixty. For some, this is a tearing ordeal, movingly attested by Hardy's *Apology*; others adapt by finding an activity in which they can exercise the faculties they still possess.

As with many scientists, the life of the mathematician is governed by an insatiable curiosity, a desire to solve the problems under study that borders on passion and can lead him to abstract himself almost completely from the surrounding reality; the absent-mindedness or oddities of famous mathematicians have no other origin. This is because the discovery of a proof is generally achieved only after periods of intense and sustained concentration, which may recur for months or even years before the sought-after result is reached. Gauss himself acknowledged that he had searched for the sign of an algebraic expression for several years, and Poincaré, when asked how he had arrived at his discoveries, replied, "by thinking about them often"; he moreover described in detail the course of his reflections and attempts that led him to one of his finest results, the discovery of the "Fuchsian" functions [10].

The possibility of having enough time to devote to one's work is therefore what a mathematician seeks above all, and this is why, since the nineteenth century, teaching careers in universities or technical schools—where the number of teaching hours is relatively small and vacations are long—have been preferred. The level of remuneration comes only second, and one has even seen recently, in the United States among other places, mathematicians leave lucrative positions in industry in order to return to the university, at the cost of a substantial reduction in salary.

It is in fact only in recent times that the number of teaching posts in universities has become sufficiently large; before 1940 it was very limited (fewer than 100 in France), and until 1920 mathematicians of the stature of Kummer, Weierstrass, Grassmann, Killing, and Montel had to content themselves, for all or part of their careers, with being secondary-school teachers—a situation that long persisted in small countries with only a few universities.

Before the nineteenth century, the opportunities for a mathematician to work were even more precarious, and in the absence of personal wealth, a patron, or an academy that could ensure him a decent material existence, he had little choice but to become an astronomer or a geodesist (like Gauss, who devoted a considerable amount of time to these professions). It is no doubt this circumstance that explains the very small number of talented mathematicians before 1800.

This same necessity of devoting long hours of reflection to the problems they seek to solve almost automatically excludes the possibility of simultaneously carrying out absorbing tasks in other fields (such as administration) and pursuing serious scientific work; the case of Fourier, prefect of the Isère at the time when he was creating the theory of heat, is probably unique. If mathematicians have at times been known to occupy high administrative or governmental positions, it is because they had more or less abandoned their research for the duration of those functions. Moreover, as Hardy ironically remarks [8, p. 14], "the careers of mathematicians, once they have left mathematics, are not particularly encouraging".

As for the behaviour of mathematicians, if one sets aside their absent-mindedness and the lack of "practical sense" with which many of them are afflicted, one may say that there is scarcely any difference between them and the members of a population of comparable size taken at random. One finds among them the same variety of characters, virtues, and vices. If they sometimes tend to take pride in possessing a talent that is not very widespread, they should remember that many other skills are accessible only to a small number of people; there are undoubtedly far fewer tightrope walkers capable of dancing on a rope, or chess players who can play blindfold. All the same, as Hardy says [8, p. 18], what a mathematician does "possesses a certain character of permanence", and one has not lived in vain when one has added a small stone to the pyramid of human knowledge.

In any case, mathematicians must convince themselves that their talent confers on them no particular competence outside their own field. Some who believed that their prestige among their colleagues would allow them to reform society have learned this to their cost.

Mathematicians' opinions in matters of religion or politics are, moreover, very diverse: Cauchy was a bigot, whereas Hardy was a curious atheist for whom God was a personal enemy; Gauss was very conservative, but Galois a fiery revolutionary. An extreme and rather distressing case is that of the young German mathematician O. Teichmüller, who may have possessed a genius comparable to that of Galois; but he was a fanatical Nazi who had contributed to the expulsion of his Jewish professors; having gone to the Russian front as a member of the SS, he never returned.

Most mathematicians lead the lives of respectable bourgeois, little concerned with the tumults of the world, desiring neither power nor wealth and content with modest comfort. Without actively seeking honours, most of them are nevertheless not insensitive to them. Unlike those of many artists, their lives are rarely shaken by emotional tempests; devotees of sensation and romance will find little to glean there, and must content themselves with the more or less romanticised biographies of Galois, Sonia Kovalevskaya, or Ramanujan.

3. The work of mathematicians and the mathematical community

The experimental sciences are carried out in laboratories, where increasingly large teams are required to handle the instruments and analyse the results. To do mathematical research, only paper and a good library are necessary. Teamwork of the kind practised in the experimental sciences is therefore rather rare in mathematics, most mathematicians finding it difficult to think seriously except in silence and solitude. Collaborative work, which is nevertheless fairly common, most often consists in confrontations of what each collaborator has been able to obtain independently, with each then benefiting from the ideas of the others in order to make progress from new starting points. The most famous example of prolonged collaboration between mathematicians is that of Hardy and Littlewood: one lived in Oxford, the other in Cambridge; they saw each other only rarely, and their joint work was carried out entirely by correspondence.

But although the majority of publications are individual, few mathematicians, like Grassmann, Hensel, or Élie Cartan, are able to work for long periods and fruitfully in almost complete isolation (due, in these three cases, to the novelty of their ideas, which were not understood by their contemporaries). Most become discouraged if they lack the means to communicate fairly frequently with their colleagues in the hope of being understood—especially since the abstract nature of their research makes the exchange of ideas with non-mathematicians difficult.

Until the middle of the seventeenth century, the only means of communication were private correspondence (sometimes centralized by volunteer letter-writers, such as Mersenne or Collins), personal visits, and occasionally the printing of a work at the author's own expense (unless a patron agreed to bear the cost). The first publications regularly supported by academies appeared around 1660, but their number remained small, and the few scientific journals published before 1820 were devoted to science without distinction of speciality. The first specialized mathematical journals to cross national boundaries were *Crelle's Journal*, founded in 1826, followed shortly thereafter by *Liouville's Journal* in 1835.

But at that point linguistic barriers begin to arise. In the seventeenth century, almost all scientific publications were written in Latin (Descartes' *Geometry*, together with a few writings by Pascal, being the only notable exceptions). This tradition began to decline in the eighteenth century: French mathematicians wrote only in their own language, and abroad some instructional works were written in the author's native tongue. After Gauss, the use of Latin rapidly disappeared from scientific publications; this inevitably slowed the diffusion of new results, especially in France, where the study of modern languages was scarcely practised before the end of the nineteenth century. A rather ridiculous example was the decision by the Paris Academy of Sciences, in 1880, to set as a prize problem a question in number theory that had been solved more than twenty years earlier by H. J. S. Smith; but apparently Hermite and Jordan did not read English. Nowadays, as is well known, English is on the way to becoming a universal language of science.

Throughout the nineteenth century, the number of mathematical journals continued to increase steadily, a growth that accelerated after 1920 with the rise in the number of countries in which mathematical studies were developing, culminating since 1950 in a veritable explosion: today there exist about 500 periodical mathematical publications worldwide. In order not to be drowned in this ocean of information, journals devoted solely to listing and briefly analysing other publications have been created since the last third of the nineteenth century. The most widespread of these periodicals, the American *Mathematical Reviews*, now run to nearly 4000 pages per year (with an average of five to ten articles reviewed on each page).

Parallel to the multiplication of periodicals has been that of didactic works, often grouped into series of monographs (sometimes quite specialized); thus it is rare today for a new theory to wait more than ten years before becoming the subject of expository treatments.

In German universities, around the middle of the nineteenth century, the practice of "seminars" was born: under the direction of one or more professors, various mathematicians—local or visiting, often including doctoral students—analyse the state of a question or review the most striking recent developments, in periodic sessions extending over the academic year. After 1920, this institution gradually spread throughout the world; the lectures given there are often circulated in mimeographed form and can thus reach a wider audience; the same is true of specialized courses taught in many universities and intended for advanced students.

But it has long been recognized that in science verbal exchanges are often more fruitful than the reading of papers. The travel of students from one university to another has been traditional since the Middle Ages, and this tradition has continued into our own time, notably in Germany. Moreover, invitations for professors to come and work and teach at universities other than their own have become fairly frequent.

This need for personal contact has been institutionalized since 1897 in the International Congresses of Mathematicians, which since 1900 have met every four years (with two interruptions caused by the world wars). The steadily increasing number of participants in these congresses has somewhat reduced their effectiveness, and since 1935 more restricted meetings have multiplied: colloquia, symposia, working groups, summer schools, and so on, where a more or less broad spectrum of specialists meet, confront their recent discoveries, and discuss the problems on the agenda.

4. Masters and schools

The spirit of competition, rankings, the race for records and prizes have never been as exacerbated as in our own time; this is obvious in sports, and it extends to intellectual activities such as chess or bridge, and even to many others that are more or less ridiculous.

Nevertheless, in recent times one has heard loudly proclaimed the idea that, in matters of intellectual creation, all human beings have the same capacities, and that the glaring inequalities observed in this respect are due solely to education, which has favoured some more than others. This is a curious doctrine, which would require the brain to have a physiology different from that of other organs, and which deserves nothing more than a shrug of the shoulders when one thinks of all the children of princes or millionaires who have remained incurably stupid despite the care taken to provide them with the best tutors.

One must therefore admit that, as in all disciplines, the value of a mathematician varies greatly from one individual to another. In "developed" countries, the teaching of mathematics is ensured by a large body of instructors, most of whom have had to obtain a doctorate (or an equivalent degree) by carrying out an original piece of research, but who are far from being equally gifted for research, for lack of creative imagination.

The great majority of these works are in fact what are called trivial—that is, they confine themselves to drawing a few easy consequences from well-known principles. Often the theses of these mathematicians are not even published; inspired by a "patron", they reflect the latter's ideas more than their own. Thus, once left to themselves, they may publish from time to time an article directly related to their thesis, and then very early cease all original production. Their importance is nonetheless unquestionable: in addition to the leading social role they play in the education of an entire country's scientific elite, it is they who, in the first years of university, can discern particularly gifted students who will become the mathematicians of the next generation. If they know how to keep abreast of the movement of science and to enrich their teaching accordingly, they will awaken hesitant vocations and guide them toward those colleagues whose task it is to direct the first steps of future researchers.

At a higher level there is a much smaller category (especially in countries that, like the United States, require an enormous teaching staff): that of mathematicians capable of going beyond their doctoral work, or even of embarking on entirely different paths. They often remain active in research for some thirty years and publish several dozen original papers. In "developed" countries, one may say that, on average, there emerges one such mathematician per year for every ten million inhabitants; this amounts to about 150 active creative mathematicians in a country the size of France, and about 600 in the United States or the U.S.S.R. These alone are able to take on, with real benefit, the so-called "third-cycle" teaching through which new ideas spread, and to advise effectively the young mathematicians who are entering research.

Finally, there are the great innovators, whose ideas bend the whole science of their time and sometimes resonate for more than a century. But, as Einstein said to Paul Valéry, "an idea is so rare!" (obviously a great idea, in Einstein's mind). One counts about half a dozen geniuses of this kind in the eighteenth century, some thirty in the nineteenth; today, one may expect that one or two appear each year worldwide. Working in contact with these great mathematicians is an exhilarating and enriching experience; Hardy saw in his association with Littlewood and Ramanujan “the central event of his life” [8, p. 58], and I can say the same of my collaboration with the Bourbaki group.

There is no Nobel Prize for mathematicians. But since 1936, the International Congress of Mathematicians, which meets every four years, has awarded at each of its sessions two, three, or four medals, known as the “Fields Medals”, to mathematicians in principle under the age of forty, whose work is judged the most remarkable by an international committee. Up to now there have been ten laureates from the United States, five French, four English, three Scandinavians, two Russians, two Japanese, one German, one Belgian, one Chinese, and one Italian. One can always criticize the way these awards are distributed, or even their very principle. But I believe no one can dispute the merit of the laureates, all of whom have had at least one great idea in the course of their careers; one should moreover add to this list some fifteen mathematicians of comparable stature whom the limited number of medals has not made it possible to honour.

Questions of nationality are no more important for mathematicians than for researchers in the other sciences; once linguistic barriers have been overcome (most often through the use of English), a French mathematician will feel far closer, in terms of conceptions and methods, to a Chinese mathematician than to an engineer from his own country.

Natural associations among mathematicians are formed far less by country than by schools; while many of these are centred in a particular country, there are countries in which several schools flourish, and schools whose influence extends well beyond national borders. These are not, of course, entities sharply defined in time or space, but groupings characterized by a continuous tradition, common masters, and preferred subjects or methods.

Before 1800, mathematicians, few in number and widely scattered, did not really have students in the strict sense, although since the middle of the seventeenth century communication among them had been frequent and mathematical research had experienced almost uninterrupted progress. Chronologically speaking, the first mathematical school was formed in Paris after the Revolution, thanks to the founding of the École Polytechnique, which provided a nursery of mathematicians until about 1880; after that date, the École Normale Supérieure took over this role. In Germany, Gauss was still an isolated figure, but subsequent generations created centres of mathematical research in several universities, the most important being Berlin and Göttingen. In England, mathematical research in the universities had fallen into lethargy after 1780; it revived only around 1830 with the Cambridge school, particularly fertile in the nineteenth century in logic and algebra, and which has shone without interruption down to our own day in mathematical physics. Italy had only a few isolated mathematicians from 1700 to 1850, but several active schools then developed there, notably in algebraic geometry, differential geometry, and functional analysis.

In the twentieth century, because of wars and revolutions, mathematical schools underwent many vicissitudes. There is first of all a phenomenon that is difficult to explain, given the very diverse political and social conditions in the countries where it occurred. After the war of 1914–1918, a whole constellation of first-rate mathematicians suddenly appeared in the U.S.S.R. and in Poland, countries that until then had known only a small number of scholars who had achieved international renown. The same phenomenon occurred in Japan after the war of 1939–1945, although, because of a rigid and impoverished university system, the Japanese school lost some of its best representatives to the United States.

The reasons that can hinder the progress of mathematical research are easier to understand. In France, the scientific youth had been bled white by the slaughter of 1914–1918, and the French school had withdrawn into itself around its older representatives. Germany, on the contrary, had been better able to preserve the lives of its scholars and to maintain its great traditions of universality, thereby ensuring an exceptional influence for its mathematical schools; it was there that students from many countries came to be trained, notably young French mathematicians eager to reconnect with traditions that had been forgotten at home since the death of Poincaré in 1912. But after 1933, the development of the German and Italian schools was brutally broken by fascism; it was only after 1950 that they were reconstituted, this time under the influence (by a curious reversal of roles) of French mathematicians of the "Bourbakist" tendency. In Poland, it was physically that the mathematical schools were annihilated, since half of the mathematicians there were massacred by the Nazis; they did not begin to regain their importance until after 1970. As for the English and Russian schools, they were able to pass through the turmoil without major damage.

Finally, the formation of mathematical schools in the United States deserves closer attention, for it illustrates the difficulty of establishing a research tradition where none previously existed, even in a country rich in people and resources. Until about 1870, no creative mathematician of any renown had appeared there: the development of a continent left little room for abstract speculation. After 1880, the first efforts to create mathematical centres consisted in inviting a few European professors (mainly English) to teach in universities, in founding periodicals devoted to mathematics, and in sending gifted young students to become acquainted with European mathematics. These efforts met with success from about 1900 onward; a first school attracted international attention in Chicago, followed between 1915 and 1930 by Harvard and Princeton. An unexpected reinforcement came from the mass emigration of European mathematicians driven out by totalitarian regimes. It was they who powerfully helped, after 1940, to bring about the emergence of today's very brilliant *native* American schools, which have taken a leading position through spectacular discoveries, notably in group theory, algebraic topology, and differential topology.

This slow development of mathematical schools has been repeated in India and in Latin America, with greater difficulty, owing to economic and political upheavals. China, after sixty years of turmoil, also seems to be entering this process, which could soon lead to the emergence of mathematicians of great talent, judging by those who were able to pursue their careers outside the country. But many developing countries have not yet succeeded in founding a lasting school. Moreover, here as in all the sciences, a "critical mass" must be reached; except in exceptional cases (the Scandinavian countries since 1900, Hungary between 1900 and 1940), countries that are too small cannot hope to have a genuine national school and must in one way or another attach themselves to those of their more populous neighbours.

The attraction exerted by active and numerous mathematical schools is easy to understand. Isolated, a young mathematician quickly runs the risk of becoming discouraged by the immensity of a bibliography in which he wanders without a compass. In a major centre, by listening to his teachers and senior colleagues, as well as to the foreign visitors who flock there, the apprentice researcher will much more rapidly be able to distinguish what is essential from what is secondary in the notions and results that are to constitute his basic training. He will be guided toward key works, informed about the major current problems and their methods of attack, warned against infertile domains, and sometimes inspired by unexpected connections between his own research and that of his colleagues.

Thanks to these research centres and to the network of communications that links them across the entire planet, it is unlikely that the lack of understanding from which certain innovators suffered in the past could still occur today; in fact, as soon as an important result is announced, its proof is actively examined and studied in many places during the months that follow.

Chapter II

The nature of problems in mathematics

1. "Pure" mathematics and "applied" mathematics

The terms "fundamental science" and "applied science" are commonly used in contemporary language; it is generally accepted that a science, or a part of a science, is "fundamental" if its aim is to understand phenomena, and "applied" if it seeks to master those phenomena for human purposes, whether good or bad. While some people fear the "applied" sciences because of the catastrophes they may engender, few go so far as to question the value of the "fundamental" sciences, insofar as they can be separated from the uses made of them.

One does not usually speak of "fundamental mathematics", but rather opposes "pure mathematics" to "applied mathematics", which may be interpreted as suggesting that there is a greater difference between these two parts than in other sciences. And indeed, if one analyses a little more closely what these names designate, one arrives at the conclusion that they correspond to two *very different* operations of the mind, which would be better called "mathematics" and "applications of mathematics". For when one compares, for example, molecular biology and clinical pathology, both deal with the same objects, cells, considered now in their internal structure, now in the global behaviour of an organ of the body of which they are the elements. By contrast, the "objects" with which mathematicians deal are not at all of the same nature as those of engineers or physicists; the following chapters will be largely devoted to making clear what mathematical "objects" are. But one can already say what the Greeks had clearly grasped, namely that our senses will never apprehend a number or a plane, but rather a pile of apples or a wall.

What, then, do the "applications" of mathematics consist in? It seems to me that they can be described as follows. The aim is to predict the behaviour of certain objects of the sensible world under given conditions, taking into account general laws governing that behaviour. One constructs a mathematical *model* of the situation under study by associating with the material objects being studied mathematical objects that are supposed to "represent" them, and by associating with the laws to which they are subject mathematical *relations* among these objects; the initial problem is thus translated into a mathematical problem. If one can solve it, exactly or approximately, one then translates the solution back in the opposite direction, thereby "solving" the problem that was posed.

A commonplace example today is the guidance of artificial satellites and interplanetary rockets. Such an object can be represented by a point in space, endowed with a coefficient representing its mass—that is, a given number—and whose position is specified by its three coordinates with respect to a fixed system of axes (three other numbers); finally, the instant at which it is observed is represented by the time shown by a clock, yet another number. The rocket is subject to the gravitational forces exerted by the Earth, the Moon, the Sun, and possibly other planets; in the model these forces are represented by *vectors* whose components are known functions of the rocket's coordinates. The determination of the motion is then, by virtue of the laws of dynamics, translated into a mathematical model: the solution of a system of differential equations. Mathematicians have methods that make it possible to solve such a system approximately, that is, to know at each instant the values of the rocket's coordinates with only a small error; but before the advent of computers, it would have taken months or years of "hand" calculation to obtain these values. At present, sufficiently powerful computers exist to carry out these calculations almost instantaneously, and one can therefore predict with what speed and in what direction the rocket must be launched in order to make it follow the desired trajectory.

This example is a particularly simple case, since only three numbers—called the *parameters* of the rocket—have to be computed as functions of a fourth parameter, time. In general there are many more "parameters", and the relations between them are far more complicated than Newton's law.

It was a great success of the mathematics of the seventeenth and eighteenth centuries to be able to provide mathematical models for the laws of mechanics and for the motions of the planets (what is known as "celestial mechanics") that agreed remarkably well with observation. In fact, these models constituted the first truly fruitful applications of mathematics. The theorems used there belong to what may be called the "elementary" part of the infinitesimal calculus; they aroused the admiration of contemporaries, but have now lost the brilliance of novelty and are taught in the final year of secondary education and in the first two years of university. As a result, mathematicians, as Hardy puts it, find them "dull, boring, and devoid of aesthetic value". Nevertheless, as we shall see in Chapter IV, it is rare for a mathematical problem, even once solved, not to give rise to a host of others. In the case of the differential equations of dynamics, after a period of stagnation, an entirely new branch of mathematics arose from the work of H. Poincaré beginning around 1880—what is known as the theory of "dynamical systems"—fertile in difficult and profound questions, and to which many mathematicians are now devoting themselves with greater zeal than ever.

2. Theoretical physics and mathematics

The applications of mathematics to physics expanded during the nineteenth century and, even more so, during the twentieth century, and they also changed in character. When it became necessary to construct mathematical models for the new theories of hydrodynamics, elasticity, electromagnetism, thermodynamics, and later relativity and quantum mechanics, one was confronted with mathematical problems far more formidable than those of celestial mechanics, involving in particular what are known as partial differential equations. One of the simplest examples is the equation of "vibrating strings"

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

where x is the position of a point on the string, t is time, and $u(x,t)$ is the distance by which the point of the string at position x departs from that position at time t. Although some of these models date back to the eighteenth century, the mathematicians of that period were quite incapable of solving most of the problems they posed. It required the entire effort of nineteenth-century analysts to obtain results that were significant and useful to physicists; moreover, many of these problems are still only partially solved today and continue to give rise to new research and new methods of attack.

One may therefore say without hesitation that it was the needs of physics that led mathematicians to create a new branch of their science, what is called functional analysis. This has now become a substantial part of mathematics, which continues to grow and branch out; it is noteworthy that it has sometimes been by transposing into mathematical models certain concepts specific to physicists (energy, the principle of "least action", and so on) that general methods have been obtained for solving certain equations of functional analysis.

In more recent times, the use of probability theory in statistics and in physics has likewise led to numerous studies in this area; similarly, operations research and automata theory have stimulated the study of certain parts of algebra.

3. Applications of mathematics in the classical period

Newtonian mechanics and its dazzling successes in astronomy—notably Clairaut's prediction of the return of Halley's comet, with an error of less than one month—made a deep impression on all intellectual circles in Europe, even on men of letters such as Voltaire, who understood absolutely nothing about mathematics[1]. As for the mathematicians of the eighteenth century—Clairaut, d'Alembert, and Laplace—these remarkable applications of mathematics, followed by many others, led them to consider that the essential aim of mathematical research was to provide models for mechanics and physics; any other part of mathematics that did not meet this requirement was judged futile and negligible. When Euler informed Clairaut of his results in number theory, on which he said he had been working for fourteen years, Clairaut merely replied that "this subject must be very thorny", and immediately turned to the topic that interested him, the calculation of planetary perturbations [7, p. 129].

Before the seventeenth century, the applications of mathematics were far from having the quasi-magical character that aroused admiration for Newtonian mechanics. In fact, there were only four sciences in which mathematicians had been able to construct models allowing a *rational* "explanation" of phenomena; all of them were due to Greek mathematicians, after whom no new idea appeared before the sixteenth century. These were the optics of mirrors (or "catoptrics"), statics, the equilibrium of floating bodies—which do not seem to go back earlier than the fourth century BCE—and astronomy, whose first steps are attested in the sixth century BCE.

These applications used only geometry, of which they appear as annexes[2]; lacking a mathematical tool that would have enabled them to describe arbitrary motions—a tool that became available only with the infinitesimal calculus—the Greeks, in order to "account for" the motions of the planets on the celestial sphere, started from an *a priori* idea, namely that only uniform rotations about

[1] He writes somewhere that he has never been able to understand why the sine of an angle is not proportional to the angle.

[2] To reveal his geometric theorems on areas, volumes and centres of gravity, Archimedes explains, in a short work discovered around 1900, how he breaks down his figures into "slices" which he compares using fictitious "weighings".

an axis (or, in a plane, about a point) were acceptable for heavenly bodies to which a superior "perfection" was attributed, no doubt a remnant of theological conceptions concerning the "divinity" of the stars. This led them, as is well known, to imagine a system of spheres moving relative to one another, or of "epicycles" (circles whose centre describes another circle), a system that became increasingly complicated in order to accord with observations as these grew more precise.

But before this attempt—ultimately destined to failure—the Greeks had succeeded in forming an idea, still approximate but fundamentally correct, of the shapes and dimensions of the Earth and of the two heavenly bodies whose apparent diameter is appreciable to the naked eye, the Moon and the Sun. To do this they used nothing but geometry, applied in an original way to simple observations, but chosen with rare ingenuity. I would like to summarize here these first achievements, so different in spirit from the attitude of other peoples toward the same phenomena. Even the Babylonians, whose astronomy was the most developed, confined themselves to recording the motions of the heavenly bodies (risings, settings, conjunctions, eclipses, etc.) and discovered many periodicities in them; but one finds no trace among them of a geometric model seeking to *explain* these phenomena.

One must first recall that the only measuring instruments available in Antiquity and the Middle Ages for astronomical observations were rulers of varying length, called alidades, pivoting about the centre of a graduated circle and allowing, by sighting through apertures located at the ends of the alidade, the determination of the angle between the directions of two points on the celestial sphere, with varying degrees of precision—on the order of one minute of arc for the best instruments. Any model therefore had, for them, to be reducible to angle measurements.

The first step was taken by the Pythagoreans of the sixth century BCE, who had convinced themselves that the Earth, the Moon, and the Sun were spherical bodies, and that the phases of the Moon resulted from variations in the portion illuminated by the Sun, according to the respective motions of the Sun and the Moon around the Earth, which was supposed to be immobile. They had understood that lunar eclipses were due to the Moon's passage through the cone of the Earth's shadow (fig. 1); the fact that the trace of this cone on the Moon is an arc of a circle (fig. 2) could only be explained by the spherical shape of the Earth.

Another observation of the Moon allowed them to realise that the distance TS is much greater than TL. Indeed, at first quarter the plane separating the illuminated part of the Moon from its shadowed part passes through T, and the line TL is therefore perpendicular to LS (fig. 3); hence $\frac{TL}{TS} = \cos\alpha$. The angle α is very close to 90° (it differs from it by less than one minute), and the ratio TL/TS is therefore of the order of 0.003; since sighting the Sun is difficult, the Greeks thought that α was close to 87°, and thus obtained a value of TS that was too small.

One therefore makes only a very slight error in assuming that, at all points on the Earth, the rays coming from the centre of the solar disc are all parallel. This allowed Eratosthenes, around 300 BC, to obtain a first approximation to the radius of the Earth, by the following reasoning. In Egypt, Alexandria and Syene (now Aswan) lie approximately on the same meridian (fig. 4). On the day of the summer solstice, at noon, the Sun is exactly at the zenith at Syene, and at the same hour its rays make an angle β with the vertical at Alexandria. Since the angle β is the angle between the rays joining the centre O of the Earth to Alexandria and to Syene, the meridian arc between these two cities is $d = R\beta$ (β in radians), whence $R = d/\beta$; and Eratosthenes had a rough estimate of the distance d.

From the value of R, one can deduce the distance TL from the Earth to the Moon, if one knows the ratio $A'B'/C'D'$ (fig. 2) of the radii of the Earth and the Moon; the Greeks had estimated that this ratio was close to 3 by observing the curvature of the trace of the Earth's shadow on the Moon; in fact the ratio is 3.7. If one then measures the angle ε under which the lunar disc is seen, one has (fig. 3) $D = 2r/\varepsilon$, where r is the lunar radius.

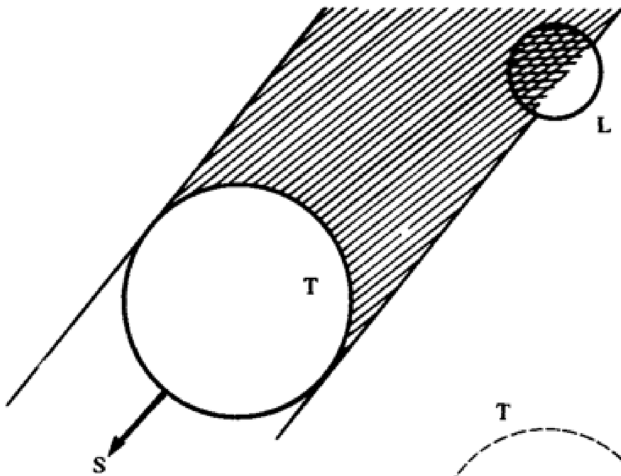


fig. 1

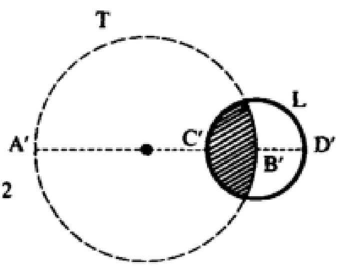


fig. 2

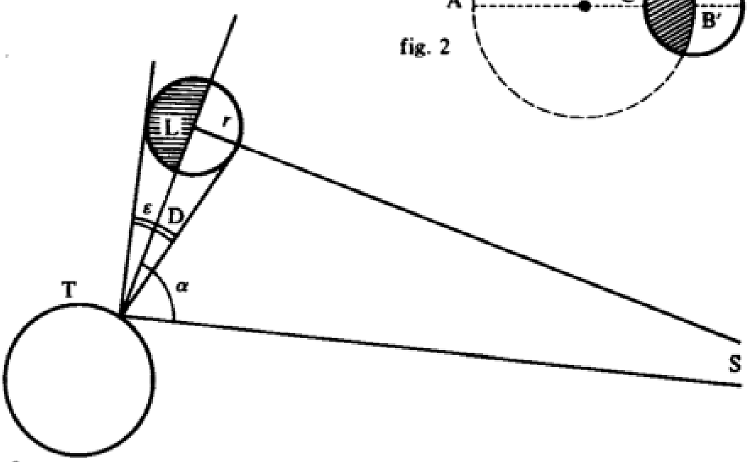


fig. 3

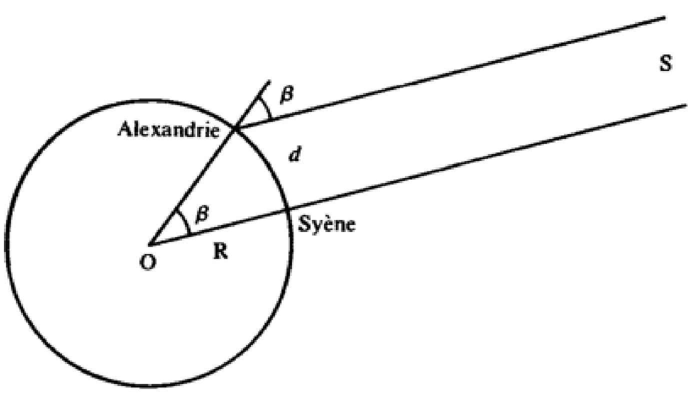


fig. 4

One should note that the applications of mathematics we have just described, just like Newton's celestial mechanics and that of his successors in the seventeenth century, concern a "fundamental" science which itself had no immediate "use" until the invention of artificial satellites. The universal admiration that this science has nevertheless always inspired is an example of a tradition uninterrupted since the Greeks: the value attributed to "the sciences that are not concerned with utility or profit", as Aristotle puts it (*Analytica Priora* 981^{b}). Here we touch upon one of the characteristics of the human being, *curiosity*, so clearly visible already in the young child, even if many adults seem to have lost it and to have sunk into routine and intellectual inertia.

Of course, the Greeks would never have thought of dissociating the properties of numbers or of geometrical figures from their applications to astronomy. On the contrary, the integers, from the Pythagoreans of the sixth century BC onwards, tended to become surrounded by a kind of mysticism, still very apparent in Plato, and which has been perpetuated down to our own day in the ramblings of "numerology".

4. Utilitarian attacks

It was in the circle around d'Alembert, with Buffon and especially Diderot, that the challenge to mathematics was to go much further than d'Alembert's own position, which made applications to the other sciences the sole aim that mathematicians ought to pursue. For Diderot, mathematics had had its day: it added nothing to experience and merely "interposed a veil between nature and the people", instead of "making philosophy popular"; one sees that the principles of the "cultural revolution" are not of yesterday! Yet Diderot, unlike Voltaire, had some acquaintance with mathematics, but he must have realised that he would never produce an original work in it. Is this what should be blamed for his attacks? As Fontenelle had already written in 1699: "Things that one does not understand are commonly called useless. It is a kind of revenge; and since mathematics and physics are generally not understood, they are declared useless."

To Diderot's credit, and to that of those who, down to our own day, have taken up his invectives against mathematics, it must be acknowledged that until the beginning of the twentieth century mechanics and theoretical physics had scarcely influenced "useful" technological inventions. The "simple machines" (lever, pulley, screw, etc.) go back to high antiquity; the steam engine preceded thermodynamics, and even the internal combustion engine and aviation were born of "tinkering" rather than of theory. It is only in our own time that technology can no longer do without the results of the "fundamental sciences", and hence of mathematics. Thus the revolutionaries who believed they could restrict the work of mathematicians to what could be made "popular" in Diderot's sense were quickly forced to beat a retreat.

In fact, it is hard to understand the animosity that erupts from time to time in pamphlets still attacking "pure" mathematics. There are many other disciplines, such as cosmology, prehistory, and archaeology, that certainly have no "utility", and yet no one rises up against them; on the contrary, public opinion is easily mobilised in their favour when a government cuts the funds allotted to them. Must one return, then, to Fontenelle's remark?

5. Fashionable dogmas

The detractors of "useless" mathematics are forced to acknowledge that, in the enormous output of contemporary mathematics, a considerable part manifestly has no conceivable "application" today. They extricate themselves by asserting emphatically that even these parts will one day become "useful". Since such an assertion refers to an indeterminate future, it is nothing other than a dogma, just as impossible to refute as to prove. The clinching argument is the fact that Kepler found the mathematical model of the motion of the planets around the Sun in the theory of conics, developed 1800 years earlier by Apollonius without the slightest intention of "applying" it to anything. More recently, two similar cases have been added: general relativity, which found its model in Riemannian geometry conceived some sixty years earlier, and the structure of atoms and of "elementary particles", in which group theory and Hilbert space play important roles.

But here are three of the most remarkable theorems of number theory:

I (Euclid, c. −300). There exist infinitely many prime numbers (or, in the original formulation, *every* integer is smaller than some prime).

II (Lagrange, 1770). *Every* integer can be written as the sum of 1, 2, 3, or 4 squares of integers (for example: $7 = 2^1 + 1^2 + 1^2 + 1^2$).

III (Vinogradov, 1937). There exists a (very large) integer N such that *every* odd integer greater than N is a prime number or the sum of three odd prime numbers[3].

Each of these results states a property belonging to an infinity of integers; if one were to weaken the statements by restricting them, for example, to integers less than 10^{1000}, one would still have theorems, but for most mathematicians they would arouse little interest. Now it seems scarcely conceivable that a physical theory should ever need to consider an infinite set of integers.

There are other opinions that astonish mathematicians; they are put forward by certain historians of science. They do not consider sufficient those works on the history of mathematics that describe past ideas while trying to understand their sequence and the influences they exerted on one another; according to them, one should also "explain" why mathematicians chose one or another direction of research, and how they arrived at their results. I confess that I do not understand what this can mean: the activity of a creative brain has never had a rational "explanation", in mathematics any more than elsewhere. All that is known is that it involves periods—sometimes long—of unsuccessful attempts, followed by sudden "illuminations", and by a "shaping" of what they have revealed; all this was very well described by Poincaré, on the basis of his own experience [10].

As for the origin of the problems a mathematician wishes to solve, it must almost always be sought in their relations with other mathematicians: the reading of classical works or textbooks, a conversation, an exchange of letters, or the attendance at a lecture. But this does not suffice for those historians of science

[3] An example of such a decomposition is $33 = 23 + 7 + 3 = 11 + 11 + 11$. It is conjectured that this decomposition holds for every odd composite integer; one would therefore have to verify it for all integers less than N, but N is too large for this to be possible, even with the most powerful computers.

If any reader were to imagine that such statements are rare in mathematics, I would urge them to consult the 1289 pages of L. E. Dickson's *History of the theory of numbers* (2 vols., Carnegie Institution, Washington, D.C., 1920); they describe exhaustively all papers relating to number theory from its origins up to 1918. A work of comparable size would be required to give an account of everything that has been done in this branch of mathematics since then.

who criticise the conception mathematicians have of their own discipline; they claim to uncover there their universal "explanation", namely the influence of the surrounding social environment. It is well known that this is a dogma proclaimed by many intellectuals, out of a desire to minimise the contribution and originality of individuals and to combat what they call "elitism". I am not competent to judge the value of this dogma in other sciences, although I fail to see how the societies in which they lived could have influenced the discoveries of Newton or Einstein. But in mathematics, apart from those parts that serve as models for other sciences, the dogma seems to me perfectly absurd, in view of everything that is known about the way mathematicians of past centuries worked and about the behaviour of their contemporaries. How can one conceive that social environments as diverse as Ptolemaic Alexandria for Euclid, the Paris of Louis XV or the Berlin of Frederick II for Lagrange, or Stalin's U.S.S.R. for Vinogradov, could have had anything in common with the number-theoretic theorems cited above?

There is another side to this dogma, which claims to see in the motivations of a scientific researcher the desire to be "useful" to society. I shall content myself with quoting what Hardy says about this [8, p. 20]: "If a mathematician, a chemist, or even a biologist were to tell me that the motive force of his work had been the desire to contribute to the good of humanity, I should not believe him, or if I did believe him, I should not think any better of him for it." I am convinced that this is what most mathematicians think, even if they do not say so openly for fear of being reviled by the reigning "intelligentsia".

6. Conclusions

Let us summarise this long discussion in a few points.

I) There is an important part of mathematics that arose in order to provide models for the other sciences, and there is no question of minimising it. But it certainly does not constitute more than 30 to 40 per cent of contemporary mathematics as a whole, as one can easily see by browsing the monthly publication *Mathematical Reviews*, which gives brief analyses of everything that is published in mathematics and in the most important of its applications.

II) One may say with Hardy [8, p. 20] that the principal reason that drives a mathematician to do research is intellectual curiosity, the attraction of riddles, the need to know the truth: "The problem is there, you *must* solve it", says Hilbert. We shall see in Chapter IV how this insatiable passion has led to the extraordinary proliferation of questions that mathematicians never cease to ask themselves. There then come, of course, motives that are in no way dishonourable and that they share with many others, such as the desire to leave behind something lasting and, no doubt also, the ambition to be esteemed in one's lifetime for the work accomplished.

III) The mathematician knows perfectly well that this esteem can come only from their peers. They live in a closed milieu, with almost no communication with the outside world, and it is from this milieu that their problems arise, except for those parts that serve as models for the other sciences.

IV) However high the opinion a mathematician may have of their own work, it is the judgement of their peers that will determine its value. It is evident that such work must be "nontrivial"; it will be all the more admired if its difficulty was great and if it had been the object of numerous previous unsuccessful attempts. Beyond that, there is no criterion of evaluation that is invariant from one period to another or from one mathematician to another. People speak of “generality” and of “depth”, but these words do not always have the same meaning for those who use them; there are also fashions and enthusiasms, which temporarily enhance certain parts of mathematics at the expense of others. These divergences of judgement recall the disputes provoked by works of art, and indeed mathematicians often discuss among themselves the greater or lesser “beauty” that they attribute to a theorem. This does not fail to surprise practitioners of the other sciences: for them the only criterion is the “truth” of a theory or a formula, that is, how well it accounts for the observed phenomena. In mathematics, all results are “true”, in the sense that they have been proved according to the accepted logical rules (cf. Chapters III and VI); an unproved assertion is not part of mathematics. Other criteria are therefore needed to evaluate a mathematical work, and they can only be subjective—which leads some to say that mathematics is much more an art than a science.

Chapter III

Objects and methods of classical mathematics

While all ancient civilisations, for the needs of everyday life, had to develop procedures for arithmetical calculation and the measurement of quantities, only the Greeks, from the sixth century BC onwards, thought to analyse the logical chains underlying these procedures and thus created an entirely new mode of thought. In this chapter we shall try to bring out the essential characteristics of Greek mathematics and the unsuspected developments of extraordinary fertility that mathematicians gave it between the Renaissance and the end of the eighteenth century.

In §2 and 3 we highlight the two fundamental characteristics of Greek mathematics:

1) The idea of *proof*, by a succession of logical inferences starting from propositions that are not proved—axioms and postulates. It must be stressed that the implementation of this idea was possible only thanks to the virtuosity acquired in the handling of logic within the Greek philosophical schools; a particularly striking example is provided by the principle of "proof by contradiction", a refined tool of logicians that became one of the pillars of mathematical reasoning.
2) The objects with which mathematicians deal bear the same names as those that enter into practical calculations: numbers, geometrical figures, and magnitudes. But as early as the time of Plato, mathematicians were aware that under these names they were reasoning about beings of an entirely different nature—*immaterial* beings obtained "by abstraction" from objects accessible to our senses, but which are only their "images".

In §4, taking the notion of a geometrical figure, we show how the properties attributed by the axioms to the "abstract" objects of geometry make them profoundly different from their "images", and the difficulties that result in *defining* these objects by means of an appropriate vocabulary. In order to fix ideas, we do not pause here to follow in detail the historical vicissitudes of these conceptions, which we shall encounter again in Chapter VI; instead, we move directly to the approach by which Pasch and Hilbert, at the end of the nineteenth century, by taking up Euclid's axiomatics in the same spirit but by filling in its gaps, succeeded once and for all in escaping these difficulties, by positing that it is the axioms to which mathematical objects are subject that *define* those objects.

Section §5 is similarly devoted to the mathematical objects whose "images" are the numbers and magnitudes of sensible reality. The "abstract" character of the notion of the integer has always been present in Greek arithmetic, and Euclid's exposition of divisibility of integers and of prime numbers is, in substance, still the one that is taught today; although it is not cast in the form of an axiomatic theory—unlike geometry—very little would need to be added to achieve this (Chapter VI).

By contrast, the discovery of incommensurable magnitudes brought about a crisis in the Greek conception of the measurement of magnitudes. It seems in fact that previously the Pythagoreans had always assumed that, once a unit had been chosen for a given kind of magnitude, every magnitude of that kind had a "common measure" with the unit—we would say that its measure is a rational number. To overcome this difficulty, the Greeks created new mathematical objects, the *ratios* of magnitudes of the same kind, which they defined axiomatically in such a way that the ratios of magnitudes of a given kind to a unit chosen once and for all form a *part* of what we call the set of *positive real numbers*; this part contains the rational numbers and certain irrational numbers, but the texts do not allow it to be delimited precisely.

No doubt for philosophical reasons, the mathematicians of Plato's school imposed taboos on themselves in the handling of the three kinds of geometrical magnitudes: lengths, areas, and volumes. For example, one could not add the measures of a length and of an area, and the product of the measures of two (respectively three) lengths was the measure of an area (respectively a volume), and not the measure of a length. While geometry could accommodate these restrictions, they made impossible an algebraic calculation of the kind we practise on the real numbers; it required the authority of Descartes to secure the adoption of this kind of calculation, which had already been suggested by certain mathematicians of earlier centuries. From that point on, the "ratios" of magnitudes of the same kind were identified with real numbers, without it being necessary to specify the kind of magnitude under consideration.

In §7 and 8 we show how, combined with the invention of convenient notations during the Middle Ages and the Renaissance, this reform made possible not only the development of algebra, but also the decisive creation of the *method of coordinates*. This provided, on the one hand, an algebraic model of Euclidean geometry and, on the other, made it possible to conceive the general notion of a real function of a real variable, which had remained foreign to the Greeks.

Finally, §6 and 9 deal with two of the most fundamental notions in mathematics: that of *approximation*, and that of *limit* which derives from it. Greek mathematicians usually solved algebraic problems by means of geometrical "constructions": Euclid, for example, gives the construction of the square root of a "ratio" by the intersection of a circle and a straight line, and Menaechmus similarly constructed a cube root by the intersection of two conics. But one also finds in Euclid another idea for defining the measure of the area of a non-polygonal plane figure: he encloses it between two sequences of polygons whose difference in area tends to zero. This idea was used repeatedly by Archimedes; it was generalised in the seventeenth century and made it possible, for example, to prove the existence of nth roots for $n \geq 4$, something the Greeks could not achieve by geometrical constructions. To justify these procedures it is, of course, necessary to introduce an axiom that remained implicit until the nineteenth century; it was finally made explicit by Cauchy under the name of the "principle of nested intervals". This axiom, together with those of Euclid, completes the axiomatic definition of the set of *all* real numbers; it provides a solid foundation for the infinitesimal calculus, invented in the seventeenth century, which was to become the most powerful tool of pure mathematics and of its applications.

1. The birth of pre-mathematical notions

In contemporary society, the notions of number and of the measurement of magnitudes are acquired in childhood; by the age of twelve or thirteen they appear so "natural" that their use becomes a mental automatism. But Piaget's experiments have shown that before the age of twelve, although the "arbitrary" integer is conceived very early, the comparison of two magnitudes of the same kind still raises conceptual difficulties for certain kinds of magnitudes, such as volume or weight. Ethnologists, moreover, have encountered primitive societies in which integers beyond a few units have no name and therefore, *a fortiori*, cannot be objects of calculation.

The texts coming from the earliest Eastern civilisations, in Egypt or Babylonia, are too fragmentary to allow us to follow the way in which a rudimentary arithmetic and geometry may have been constituted; they already appear fully developed by the second millennium BC. Of course, these are not abstract speculations, but practical recipes, transmitted by castes of specialised scribes, and intended to deal with the practical problems posed by an already highly structured agrarian society: exchanges, dues, disputes, partitions[1].

It is not necessary for our purposes to go into the details of the problems solved in the documents that have come down to us. Let us merely say that in arithmetic they attest knowledge of fractions, of arithmetic progressions, perhaps of geometric progressions, as well as of the "rule of three". Among the Babylonians one even finds the solution of problems equivalent to a quadratic equation; for example, one tablet shows the figure of a square with the following text: "I have added the side of the square to its area and I obtain 3/4; what is the side?" This is the equation that we write

$$x^2 + x = \frac{3}{4}$$

and which the scribe solves in exactly the same way as we do: he adds $\frac{1}{4}$ to both sides and finds that the square of $x + \frac{1}{2}$ is 1, from which he deduces $x = \frac{1}{2}$.

In plane geometry, figures such as rectangles, triangles, trapeziums, right angles, and circles are known, perhaps in connection with the use of tools such as the potter's wheel, the surveyor's rope, and the mason's square. The idea of similarity is attested among the Babylonians: one text states that in a staircase, the ratio of the height to the width of a step is the same as that of the total height of the staircase to its horizontal projection (fig. 5). Moreover, the Greeks attribute to Thales a method for measuring the height of a pyramid, no doubt already known to the Egyptians: one observes the length of its shadow, and the ratio of the height to this shadow is the same as that of the height of a stick to the length of its shadow (fig. 6). As for geometry in space, the surviving monuments testify to empirical knowledge among architects and masons that is difficult to pin down precisely.

The concepts involved concern only *concrete* objects: the enumeration of objects in a collection, the measurement of magnitudes that can be added and subtracted, such as length, area, volume, weight, and angle, for each of which a unit is chosen, often together with its multiples or submultiples.

[1] Greek authors emphasise that, as the flooding of the Nile altered the shape of the fields, it was necessary to call upon a skilled craftsman, a scribe who knew the formulas, so that after the flood everyone could find a plot of land equivalent to the one they had owned before.

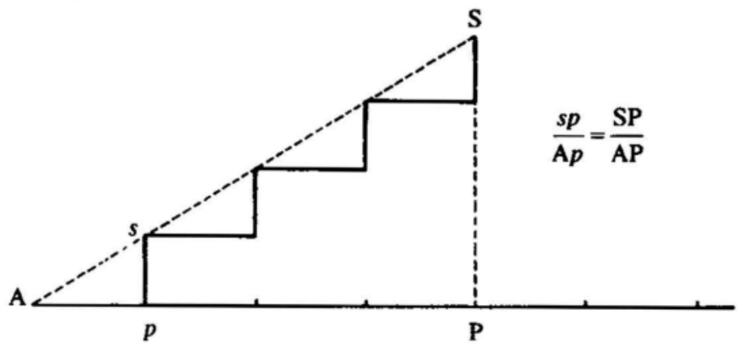


fig. 5

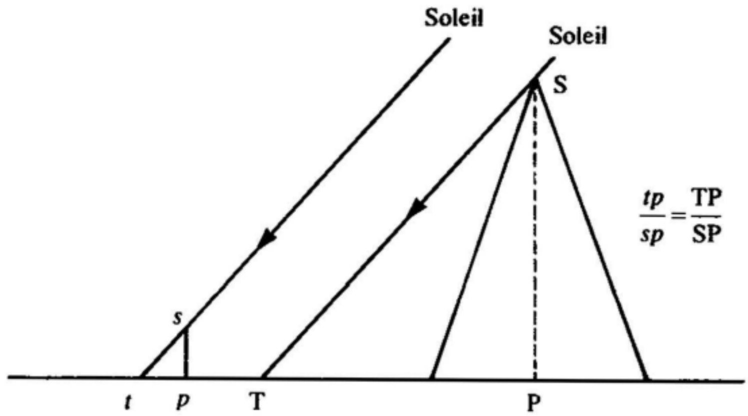


fig. 6

The procedures deal with examples in which the data are *explicitly* given; they are methods of calculation without justification, such as the evaluation of areas whose shape and dimensions are known—for example, the isosceles or right-angled triangle, the trapezium, or the circle[2]. There is, of course, no *formula* in the sense in which we understand it, applying to arbitrary and unspecified data; the generality of a method of calculation can only be inferred when a series of examples is given in which the data vary.

2. The idea of proof

No document from any ancient civilisation earlier than the fragments of Greek authors of the seventh and sixth centuries BC allows us to detect, outside those fragments, examples of what we call *logical deductions*; that is, sequences of *inferences*—later codified as syllogisms[3]—which *compel* an interlocutor to assent to an assertion Q once they have assented to another assertion P[4]. It is known that, from the fifth century onwards, Greek thinkers had become masters in organising discourse into chains of logical deductions, as is shown by the fragments of the works of the Sophists, as well as by Plato's dialogues. They had discovered that such forms of reasoning could take as their object any human activity and, in particular, the arithmetical and geometrical procedures most of which probably came from Egyptian and Babylonian civilisations. It was these procedures that were to be linked together by *proofs* into theorems.

[2]For some polygons, these formulas only give an approximate value for the area; however, for circles, some Egyptian texts give the correct approximate value of 3.16 for π.

[3]The various types of syllogisms devised by logicians have never been used by mathematicians.

[4]The use of letters to designate unspecified propositions dates back to Aristotle.

Tradition, known only through late texts, traces the first theorems back to Thales, at the end of the seventh century BC, but their proofs—if they ever existed—are unknown to us. By contrast, it is generally accepted that the theorems of the Pythagoreans—including, of course, the so-called "theorem of Pythagoras"—were accompanied by proofs, although here again their nature is unknown. The first texts that actually contain proofs are found only in Plato and Aristotle.

In the dialogue *Meno*, Socrates wishes to lead a young, uneducated slave to discover how to construct a square whose area is double that of a given square $ABCD$ (fig. 7) The boy first replies that it suffices to double the side, and Socrates shows him that the area of the new square would then be not twice, but four times that of $ABCD$. He then leads him to construct the square $A'B'C'D'$ whose side is equal to the diagonal of $ABCD$, and he shows that this square has the required property; this is the special case of the theorem of Pythagoras for the isosceles right-angled triangle.

The demonstration consists in pointing out that the square $ABCD$ is decomposed into four equal triangles by its diagonals, and that each of them—for example OAB—is equal to the triangle $A'AB$ constructed on the other side of its hypotenuse; hence in all there are eight triangles equal to OAB, whose union is $A'B'C'D'$. Of course, later on, in Euclid, the equality of the triangles OAB and $A'AB$ would follow from an entire series of preceding theorems; here, Socrates merely ensures that this equality is accepted by his interlocutor.

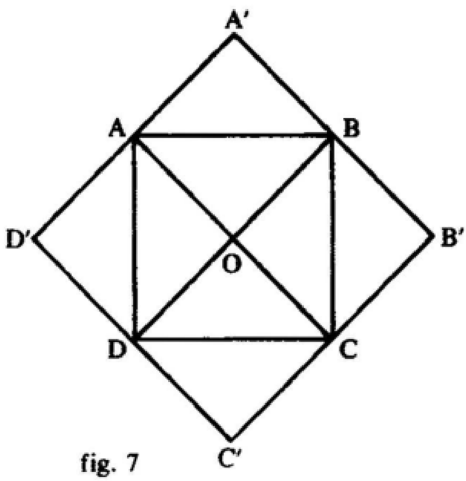


fig. 7

The second proof, reported by Aristotle as originating with the Pythagoreans, concerns the same figure of the isosceles right-angled triangle and constitutes the first known example of what is called proof by contradiction, which was to become an essential tool of mathematics[5]; it is also the first example of a statement of *impossibility*. The theorem is that, in an isosceles right-angled triangle, the ratio of the hypotenuse to one side of the right angle *cannot be a fraction* p/q (with p, q integers). Indeed, such a fraction would have the property that $(p/q)^2 = 2$, by the theorem of Pythagoras. One may suppose that the problem has been "reduced" to the case where p and q *are not both even*; otherwise one would divide them by 2 without changing the value of p/q, and by repeating this division if necessary one eventually arrives at the "reduced" case. Then $p^2 = 2q^2$; and since the square

[5]Some see it as the true birth of mathematics.

$(2n+1)^2 = 4(n^2+n)+1$ of an odd number is odd, p must necessarily be *even*, say $p = 2p'$. Then $4p'^2 = 2q^2$, so $q^2 = 2p'^2$, and q too is necessarily *even*. One has thus arrived at an absurd conclusion, and the initial hypothesis must therefore be untenable. Here, the properties that served as starting points for the inferences were the elementary properties of even and odd numbers, which seem to have been a favourite subject of the Pythagoreans and were preserved unchanged in Book IX of Euclid's *Elements*, even though by that time they had become "trivial".

3. Axioms and definitions

The way in which a proof is to proceed by a succession of inferences is very clearly described by Plato in a famous passage of the *Republic* (VI, 510 c, d):

> Those who work in geometry, calculation [...], once they have laid down by hypothesis the existence of the odd and the even, of figures, of three kinds of angles [...], proceed with regard to these notions as with things they know; using them for their purposes as hypotheses, they think it no longer necessary to give any account of them, either to themselves or to others, as though they were obvious to everyone; then, taking them as starting points and moving on from there, they finally reach, in a manner consistent with themselves, the proposition whose examination they had undertaken at the outset.

This approach has remained that of mathematicians in all periods. But reflections on the nature of the "hypotheses" of which Plato speaks (it is worth noting that he never uses the word "truth") and on the beings to which they apply have never ceased since then to preoccupy mathematicians and philosophers; this is what is called the problem of the "foundations of mathematics".

Many of Plato's dialogues are devoted to attempts to clarify "abstract" words used in ordinary language without a clear conception of what they designate, such as beauty, courage, love, piety, justice, virtue, and so on. Mathematicians, in the same way, were to have to disentangle what they meant by *figure*, *position*, *magnitude*, *quantity*, and *measure*, which are the basic notions appearing in the "hypotheses". Let us say at once that they would not fully succeed until the end of the nineteenth century, at the end of a long history whose salient episodes we shall retrace in this chapter and the following ones.

If these words designated notions drawn from sensible experience, they would raise no more problems than experience itself. But immediately after the passage from the *Republic* quoted above, Plato is careful to specify (VI, 510d) that mathematicians

> make use of visible figures, and on these figures construct their reasonings, without having these figures themselves in mind, but rather the perfect figures of which these are images [...]; they seek to see the absolute figures, objects whose vision can be possible for no one except by means of thought.

It is in fact enough to return to the scene of the *Meno*, to which Plato is clearly alluding here, to see that what Socrates states has nothing at all to do with the figures he has drawn—probably in the sand; he would be hard pressed to "prove experimentally" the equality of the triangles OAB and $A'AB$.

Plato intertwines his descriptions with considerations drawn from his theory of Ideas, which do not concern us here. But Aristotle, who does not accept this theory, nonetheless confirms the preceding passage of the *Republic*:

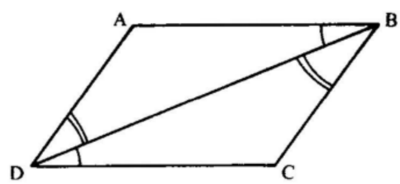
> the investigations of mathematicians (*Metaphysics* K3, 1062^{a}20–^{b}3) concern things reached by abstraction (ἐξ ἀφαιρέσεως); for they study them after eliminating all sensible qualities such as weight, lightness, hardness, and so on, retaining only quantity and continuity, the latter being conceivable in one, two, or three ways[6].

Thus the first "mathematical objects" are introduced. But understanding what these objects are, and what it is legitimate to say about them, was to give rise to innumerable difficulties right up to the end of the nineteenth century.

While we possess a substantial part of the works of Plato and Aristotle, almost nothing has come down to us from the writings of mathematicians before Euclid. It is therefore solely from Euclid's *Elements* that we can obtain fairly precise information about the conceptions of Greek mathematicians of the fifth and fourth centuries BC, even though that work appears to have been a compilation of several others from quite different periods.

The "hypotheses" that form the starting point of the process described by Plato are most often themselves derived from earlier "hypotheses". For example, Socrates' reasoning in the *Meno* is a particular case of a theorem of Euclid (Elements, Book I, Proposition 34), which proves the equality of the two triangles ABD and BCD in a parallelogram $ABCD$ (fig. 8). To do this, Euclid applies a "case of triangle congruence" (Book I, Proposition 26), noting that the side BD is common to both triangles and that the equalities of angles $\widehat{ABD} = \widehat{BDC}$ and $\widehat{ADB} = \widehat{DBC}$ follow from Book I, Proposition 29.

fig. 8



It is certain that very early on there was an attempt to "order" the theorems of geometry through a sequence of proofs, and later commentators refer to earlier Elements preceding those of Euclid, of which nothing has come down to us. But obviously this regression in "hypotheses" cannot continue indefinitely and must stop at "hypotheses" that are not proved. We know that this is indeed what we find at the beginning of Euclid's Elements, following a certain number of "definitions". It is customary today to group these "hypotheses" under the name of axioms of geometry; this seems to me to be a mistake, because they do not all relate to the same objects. Euclid was already aware of this, since he divided them into "postulates" and "common notions". The "postulates" (also called "demands") are properties of plane geometry, whereas the "common notions"—which by the end of Antiquity came to be called "axioms"—concern every kind of "magnitude" (see below, §5).

[6] A clear reference to the three dimensions of space.

4. Geometry, from Euclid to Hilbert

It is thus in Euclid that we see, for the first time, the properties of "mathematical objects" conceived in the spirit of Plato and Aristotle developed according to the deductive method. His "definitions" in Books I to VI enumerate those of these objects that belong to plane geometry: point, line, angle, circle, polygons; apart from triangles and quadrilaterals, only regular polygons are studied in any detail. Euclid leaves no doubt that these objects are not accessible to our senses: the first two "definitions" state that "a point is that which has no part" and that "a line is breadthless"[7]. What is even more characteristic are the first two "postulates":

1) that one can *join any two points* by a straight line segment; and
2) that one can *extend* a straight line segment *indefinitely* in both directions.

These properties are used constantly, but they would be absurd if they concerned "material" straight lines!

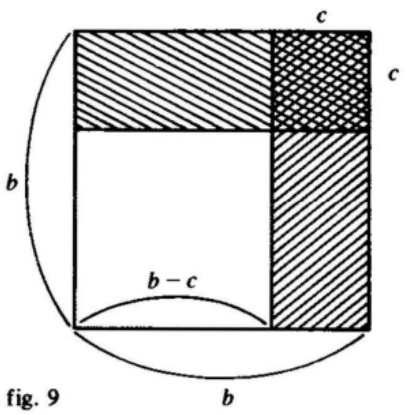


fig. 9

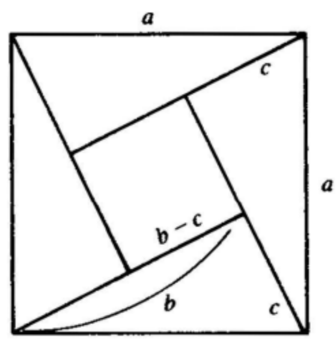


fig. 10

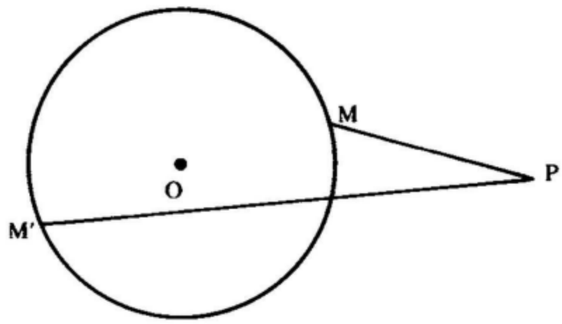
It is from his "definitions", "postulates", and "common notions" that Euclid claims to prove the succession of his theorems. What is somewhat surprising is that each of them is accompanied by a figure. One might think that this is merely an aid to following the proof more easily, and it has been said that the art of geometry consists in reasoning correctly about false figures. But one quickly realises that some of these figures play a much more essential role, not far removed from what Indian or Chinese geometers do when they content themselves with saying "see" as a proof, once the figure has been drawn[8]. For example, on several occasions (Book III, 17; Book VI, 13), Euclid assumes that a straight line having one point inside a circle meets the circle; similarly, that if a circle C has one point inside and one point outside another circle C' , then C and C' intersect (Book I, 1 and 22). None of these properties follows from his "postulates". The recourse to "visible" objects is even clearer in Book III, 8, where Euclid studies the straight-line

[7]In fact, these are nothing but unusable "pseudo-definitions". The definition of a word must involve words that have been defined previously and serves as an abbreviation; when one wishes to use it, one must "always mentally substitute the definitions in place of the defined terms", as Pascal puts it, taking up a precept already stated by Aristotle (*Topics*, VI, 4). Now, one *never* sees Euclid "substitute" these two "definitions" for the words "point" and "line"; one may therefore consider that, in his work, these words are *not defined at all*.

[8]From fig. 9, one "sees" that $(b-c)^2 = b^2 - 2bc + c^2$, and from fig. 10 one "sees" that, in a right-angled triangle with hypotenuse a and legs b, c, one has $a^2 = (b-c)^2 + 2bc = b^2 + c^2$ by fig. 9, which thus "proves" the theorem of Pythagoras.

segments joining a point outside a circle to points on the circle (fig. 11) and distinguishes on the circle the "convex circumference" (with respect to the exterior point) from the "concave circumference", notions that he would be hard pressed to define for Plato's "absolute figures". And what is one to make of the definitions given in Book XI on geometry in space, where there is talk of "surfaces described" by a straight line or a semicircle that "turn" about a straight line that remains "immobile"? One could multiply such examples; they show the difficulties that had to be overcome in order to create a vocabulary suited to the nature of objects that are only "visible to thought", and to set out their properties in accordance with that nature—that is to say, *without figures*.

fig. 11



5. Numbers and quantities

6. The idea of approximation

7. The evolution of algebra

8. The method of coordinates

9. The notion of limits and infinitesimal calculus

Chapter IV

Some problems of classical mathematics

Chapter V

New objects and new methods

1.

2.

3.

4.

5.

A. .

Chapter VI

Problems and pseudoproblems of “foundations”

1.

2.

3.

4.

5.

A. .

B. .

C. .