HODGE THEORY

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Abstract. These are notes on Hodge theory based on Benjamin Thomas Bakker's learning seminar titled *Intro to Hodge theory* at the Institute for Advanced Study for the Special Year on Arithmetic Geometry, Hodge Theory, and o-minimality (2025–26).

A Hodge structure is a certain linear algebraic datum. Importantly, the cohomology groups of any smooth projective algebraic variety come equipped with Hodge structures which encode the integrals of algebraic differential forms over topological cycles. In these lectures, we will slowly introduce the definition of a Hodge structure, and proceed to describe their parameter spaces and how they vary in algebraic families. Throughout we will emphasize how the analytic structures that show up are all definable in an o-minimal structure, and how this bridges the gap between transcendental operations (like integration) and the algebraic structures on the underlying varieties.

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1. 29 October 2025

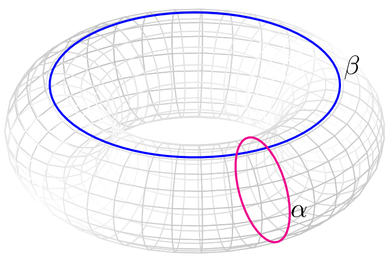
Goals:

(1) What are Hodge structures and where do they come from?
(2) What are the (parameter) moduli spaces of them?
(3) How do they vary in families?

Today we will cover (1) and start with (2). We mostly care about the o-minimal perspective of (2) and (3).

Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a smooth projective variety. For now, it suffices to think of it classically cut out by a homogeneous polynomial $\sum_{|I|=d} a_I x^I$ of degree d. Considering X as a complex manifold and its underlying topological space, we look at the singular cohomologies $H^k_{\mathrm{B}}(X,\mathbb{Z})$ (where B stands for *Betti*), which are topological invariants. The question is: can we enhance this so that it depends on the holomorphic, even the algebraic structure of X? The answer is not only we can but with profound consequences.

Example 1.1. Let $X = E \subset \mathbb{P}^2(\mathbb{C})$ be an elliptic curve (again enough to consider the form $y^2z = x^3 + axz^2 + bz^3$). As a complex manifold, $E = \mathbb{C}/\Lambda$ (with coordinate z say), where Λ is identified with $H_1(E,\mathbb{Z})$.



We can also consider the first de Rham cohomology $H^1_{C^\infty \mathrm{dR}}(E,\mathbb{C}) = \{\text{closed 1-forms}\}/\{\text{exact 1-forms}\}$ (A.4), and we have a natural map (in fact an isomorphism)

$$H^1_{C^\infty \mathrm{dR}}(E,\mathbb{C}) \to H^1_{\mathrm{B}}(E,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Z}} H^1_{\mathrm{B}}(E,\mathbb{Z}),$$

$$[\eta] \mapsto \left(\gamma \mapsto \int_\gamma \eta\right),$$

and we can write

$$[\eta] = \left(\int_\alpha \eta\right)\alpha^v + \left(\int_\beta \eta\right)\beta^v.$$

The key is that there is a unique (up to scale) holomorphic form $\mathrm{d}z$ and a subspace $\mathbb{C}[\mathrm{d}z] \hookrightarrow H^1_{\mathrm{B}}(E,\mathbb{C})$ which gives an embedding $\Lambda \hookrightarrow \mathbb{C}$, recovering E as $\mathbb{C}/\Lambda$.

Example 1.2. In general, again take X to be a smooth projective complex variety (of dimension n). Choose holomorphic coordinates $z_1,\dots,z_n$ of X, then we define a closed form η to be *of type* p,q if one can write

$$\eta = \sum f\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_p}\wedge\mathrm{d}\overline{z_{j_1}}\wedge\cdots\wedge\mathrm{d}\overline{z_{j_q}}.$$

Denote by $H^{p,q}(X)$ (where $p+q=k$) the subspace of $H^k_{C^\infty\mathrm{dR}}(X,\mathbb{C})$ of classes representable by closed forms of type p,q. It is not necessarily nonzero for a general X, but by the Hodge decomposition theorem, under our hypothesis for X (in general for a Kähler manifold),

$$H^k_{C^\infty\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

for example, going back to the previous example (1.1), one can write

$$H^1_{C^\infty\mathrm{dR}}(E,\mathbb{C}) = H^{1,0}(E)\oplus H^{0,1}(E) = \mathbb{C}[\mathrm{d}z]\oplus\mathbb{C}[\mathrm{d}\overline{z}].$$

A nonexample is the Hopf surface. Let $t\in\mathbb{C}$ with $|t|>1$ and $X = {}_{t^{\mathbb{Z}}}\backslash^{\mathbb{C}^2\backslash\{0\}}$.

Definition 1.3. A $\mathbb{Z}$*-Hodge structure* V *of weight* k is

1. $V_{\mathbb{Z}}$, a torsion free finitely generated $\mathbb{Z}$-module, with
2. a decomposition $V_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{Z}} V_{\mathbb{Z}} = \bigoplus_{p+q=k} V^{p,q}$ such that $V^{p,q} = \overline{V^{q,p}}$.

Remark 1.4. We can also have $\mathbb{Q}$-Hodge structures...

Example 1.5. Take the same X as in 1.2, now as a real manifold of dimension $2n$, and consider the middle homology $V_{\mathbb{Z}} = H^n_{\mathrm{B}}(X,\mathbb{Z}) = H_n^{\mathrm{B}}(X,\mathbb{Z})$ and the cup form for $\alpha\in H^{p,q}(X)$ and $\beta\in H^{p',q'}(X)$

$$q(\alpha,\overline{\beta}) = \int_X \alpha\wedge\overline{\beta} = \begin{cases} 0 & \text{unless } (p,q)=(p',q') \\ \text{some definiteness properties} \end{cases}$$

For example, if $X=E$ is an elliptic curve, then

$$q([\mathrm{d}z],\overline{[\mathrm{d}\overline{z}]}) = \int_E \mathrm{d}z\wedge\mathrm{d}z = 0, \qquad q(a[\mathrm{d}z],a\overline{[\mathrm{d}z]}) = |a|^2\int_E \mathrm{d}z\wedge\mathrm{d}\overline{z}\in i\mathbb{R}_{<0}.$$

Definition 1.6. Let V be a weight k $\mathbb{Z}$-Hodge structure. A *polarisation* is a $(-1)^k$-symmetric form q (it's symmetric if k is even and antisymmetric if k odd) such that

(HR1) $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$ is orthogonal for $q(u,\overline{v})$, and
(HR2) $q(u,\overline{v})|_{V^{p,q}}$ is definite of alternating sign, that is, $i^{p-q}q(u,\overline{u})>0$ for $u\in V^{p,q}$.

(where *HR* stands for *Hodge–Riemann relations*). V is polarisable if it admits a polarisation.

Example 1.7. For the same X as in 1.2, we claim each cohomology $H^k(X,\mathbb{Z})$ is a (functorially) polarisable $\mathbb{Z}$-Hodge structure. The functoriality is clear, yet in 1.5 we only defined a polarisation for $k=n$. But for $k\le n$, we can consider $h := c_1(\mathcal{O}_X(1)) = [\omega_{\mathrm{FS}}|_X]$ (where *FS* stands for *Fubini–Study metric*) and define

$$q(\alpha,\overline{\beta}) = \int_X h^{n-k}\wedge\alpha\wedge\overline{\beta},$$

which (up to sign) polarises $H^k(X,\mathbb{Z})$.

Let $V_{\mathbb{Z}}$ be a torsion free finitely generated $\mathbb{Z}$-module of fixed rank k, let q be a form and fix so-called Hodge numbers $h^{p,q}$ for each $(p,q): p+q=k$. The question is, what is the space $\mathbb{D}$ of Hodge structures V over $V_{\mathbb{Z}}$ polarised by q and satisfy $\dim V^{p,q} = h^{p,q}$?

Definition 1.8. For a decomposition $V = \bigoplus_{p+q=k} V^{p,q}$ of a k-dimensional vector space, define the *filtration*

$$F^pV := \sum_{p'\ge p} V^{p',q'}.$$

Remark 1.9. (1) A Hodge structure V is determined by its Hodge filtration F^pV; indeed,

$$V^{p,q} = F^pV\cap\overline{F^qV}.$$

(2) A filtration $F^\bullet V$ comes from a Hodge structure V if and only if

$$F^pV \cap \overline{F^{q+1}V} = 0 \ \forall p, q : p + q = k.$$

(3) (HR1) of 1.6 is equivalent to that the filtration $F^\bullet V$ of V is q-isotropic, that is,

$$(F^pV)^\perp = F^{q+1}V \ \forall p, q : p + q = k.$$

(4) Recall that (HR2) is an open condition.

Let $\check{\mathbb{D}}$ be the flag variety of q-isotropic filtrations on a fixed $\mathbb{C}$-vector space $V_\mathbb{C}$ with

$$\dim_{F^{p+1}V} F^pV = h_{p,k-p}.$$

Then

$$\check{\mathbb{D}} = \mathrm{Aut}(V_{\mathbb{C},q})/P$$

for some parabolic subgroup P, in particular a projective variety. By 1.9 (1) and (3), $\mathbb{D} \subset \check{\mathbb{D}}$, and by (2) and (4), it is an open embedding, hence $\mathbb{D}$ has a complex manifold structure.

Even better, $\mathrm{Aut}(V_{\mathbb{R},q})$ acts transitively on $\mathbb{D}$, which realises $\mathbb{D}$ as a quotient of $\mathrm{Aut}(V_{\mathbb{R},q})$ by some stabiliser group M. But $m \in M$ preserves q, so in particular preserves a definite form and hence M is compact.

Example 1.10. Let $k = 1$, $V_\mathbb{Z} = \mathbb{Z}^2$, $q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $h_{1,0} = h_{0,1} = 1$. Then any filtration is automatically q-isotropic, so $\check{\mathbb{D}} = \mathbb{P}^1(V_\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\,\mathrm{D}_2(\mathbb{C})$, and $\mathbb{D} = \mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{U}_1(\mathbb{R})$. The moduli space of Hodge structures of our hypothesis is then ${}_{\mathrm{SL}_2(\mathbb{Z})}\backslash^{\mathbb{H}}$, a modular curve.

2. 12 November 2025

Appendix A. Things Jiewei didn't know

A.1. **(Quasi-coherent) sheaves of $\mathcal{O}_X$-modules.** Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* or simply an *$\mathcal{O}_X$-module* is another sheaf $\mathcal{F}$ on X with morphisms $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ (addition) and $\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ (scalar multiplication) that make $\mathcal{F}(U)$ into a $\mathcal{O}_X(U)$-module for each open subset $U \subset X$.

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if for each $x \in X$, there is a open neighbourhood $U \ni x$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of

$$\bigoplus_{j\in J} \mathcal{O}_X|_U \to \bigoplus_{i \in I} \mathcal{O}_X|_U$$

for some I and J. Quasi-coherent sheaves of $\mathcal{O}_X$-modules form an abelian category $\mathbf{QCoh}(\mathcal{O}_X)$.

A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *locally free* if each point in X has an open neighbourhood U such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i\in I} \mathcal{O}_X|_U$ for some I (hence in particular quasi-coherent). If in addition we have $|I| = r$, we say $\mathcal{F}$ has (finite) *rank* r. Locally free sheaves are called *vector bundles*. If $r = 1$, we say $\mathcal{F}$ is *invertible*, and $\mathcal{F}$ is called a *line bundle*. The *Picard group* of X, denoted by $\mathrm{Pic}\, X$, is the set of isomorphism classes of invertible sheaves of $\mathcal{O}_X$-modules with addition being tensor product. The inverse of an invertible sheaf $\mathcal{F}$ is then its dual $\mathcal{F}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Recall that for an affine scheme $\mathrm{Spec}\, A$, its structure sheaf $\mathcal{O}_{\mathrm{Spec}\, A}$ satisfies $\mathcal{O}_{\mathrm{Spec}\, A}(D(f)) = A_f$. Now let M be an A-module, and we define the sheaf $\widetilde{M}$ by $\widetilde{M}(D(f)) = M_f$. By commutative algebra, $\widetilde{M}$ is a sheaf of $\mathcal{O}_{\mathrm{Spec}\, A}$-modules. In particular, $\widetilde{A} = \mathcal{O}_{\mathrm{Spec}\, A}$. For a scheme X, an equivalent condition for a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ to be quasi-coherent is that the restriction of $\mathcal{F}$ to each affine open $\mathrm{Spec}\, A$ is of the form $\widetilde{M}$ for some A-module M.

A.2. **Projective geometry.** A ring S is *graded* if its underlying (abelian) additive group admits a decomposition $S = \bigoplus_{d\geq 0} S_d$ such that $S_dS_e \subset S_{d+e}$. An element $a \in S$ is *homogeneous* if $a \in S_d$, and an ideal $I \lhd S$ is *homogeneous* if for each $f \in I$, after decompose f into its homogeneous parts $f = \sum_{i=0}^n f_i$, we have $f_i \in I$ for each i too. A prime ideal $\mathfrak{p} \lhd S$ is *relevant* if $\mathfrak{p} \not\supset S_+ := \bigoplus_{d\geq 1} S_d$. Denote by $\mathrm{Proj}\, S$ the set of homogeneous relevant prime ideals of S, and endow it with induced topology from $\mathrm{Spec}\, S$. Hence $D_+(f) := D(f) \cap \mathrm{Proj}\, S$ for $f \in S_+$ homogeneous form a base for the topology, and by defining $\mathcal{O}_{\mathrm{Proj}\, S}(D_+(f)) = S_f$ we have a scheme.

For a graded ring S, an S-module is *graded* if its underlying (abelian) additive group admits a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ such that $S_dM_e \subset M_{d+e}$. Let S be a graded ring and M a graded S-module, and let $n \in \mathbb{Z}$. Denote by $M(n)$ the graded S-module with $M(n)_d = M_{n+d}$, called the *twist of M by n*.

For a graded ring S and a graded S-module M, define the sheaf of $\mathcal{O}_{\operatorname{Proj} S}$-modules $\widetilde{M}$ in the same way as A.1. Then consider the structure sheaf $\mathcal{O}_{\operatorname{Proj} S}$ as $\widetilde{S(0)}$, and in general define $\mathcal{O}_{\operatorname{Proj} S}(n) \coloneqq \widetilde{S}(n)$, called *Serre's twisting sheaves*. Check that $\mathcal{O}_{\operatorname{Proj} S}(1)$ is a line bundle, and if $S = k[x_0, \ldots, x_n]$ (with the natural grading), then denote $\operatorname{Proj} S$ by $\mathbb{P}^n(k)$ and we have that $\mathcal{O}_{\mathbb{P}^n(k)}(1)$ (often simply $\mathcal{O}(1)$) generates $\operatorname{Pic} \mathbb{P}^n(k) = \mathbb{Z}$.

A.3. **Linear algebra.** For a vector space V over k, denote by $T(V)$ the tensor algebra

$$T(V) \coloneqq \bigoplus_{k=0}^{\infty} T^k V = K \oplus V \oplus (V \otimes_k V) \oplus (V \otimes_k V \otimes_k V) \oplus \cdots,$$

and by $\bigwedge V$ the *exterior algebra*

$$\bigwedge V \coloneqq T(V)/I$$

where I is the ideal generated by elements $x \otimes x,\ x \in V$. Operation on $\bigwedge V$, the *exterior product* $\wedge$, is defined by

$$\wedge : (a, b) \mapsto a \otimes b \bmod I.$$

Define the kth *exterior power* of V, denoted by $\bigwedge^k V$, to be the subspace of $\bigwedge V$ spanned by

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k,\ x_i \in V.$$

If V has finite dimension n with basis $\{e_1, \ldots, e_n\}$, then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

is a basis for $\bigwedge^k V$, and in particular $\dim \bigwedge^k V = \binom{n}{k}$. Moreover, we have a natural decomposition

$$\bigwedge V = \bigoplus_{k=0}^{n} \bigwedge^k V,$$

and in particular $\dim \bigwedge V = 2^n$.

Let V now be a finite dimensional vector space over $\mathbb{R}$. An *almost complex structure* on V is an endomorphism $I : V \to V$ such that $I^2 = -\mathrm{id}$. Such I is *compatible* with an inner product $\langle -, - \rangle$ on V if $\langle I(v), I(w) \rangle = \langle v, w \rangle\ \forall v, w \in V$.

A.4. **Differential forms.** For technical details involved in this section (e.g. well-definedness), we refer the reader to [Har77, II.8] and [Cat14, 1.2]. Let X be a complex manifold. By this, we use the sheaf-theoretic definition: a *complex manifold* is a ringed space $(X, \mathcal{O}_X)$ locally isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ where $\mathbb{C}^n$ is endowed with the usual euclidean topology, and $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf of holomorphic functions. It can unsurprisingly also be considered as a $2n$-dimensional differentiable manifold over $\mathbb{R}$.

It turns out that X is a locally ringed space, which means each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. We then define the *cotangent space* to X at x, denoted by $T_x^\vee X$, to be the $\mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbb{C}$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, and the *tangent space*, denoted by $T_x X$, to be its dual $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C})$.

In general, for a ring A, an A-algebra B and a B-module M, an A-*derivation* of B into M is a map $d : B \to M$ such that $d(a + b) = d(a) + d(b)$, $d(ab) = ad(b) + bd(a)$ and $d(a) = 0$ for all $a, b \in B$. The *module of relative differential forms* of B over A is a B-module $\Omega_{B/A}$ together with an A-derivation $d : B \to \Omega_{B/A}$ with the universal property that for any B-module M and A-derivation $d' : B \to M$, there is a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

In particular, fix $A = \mathbb{C}$ and $B = \mathcal{O}_X(U)$, then we have a B-module $\Omega_{\mathcal{O}_X(U)/\mathbb{C}}(U)$, hence in the language of A.1, we have a sheaf of $\mathcal{O}_X$-modules $\Omega_{\mathcal{O}_X(-)/\mathbb{C}}(-) \eqqcolon \Omega_X^1$, which is locally free of rank $\dim X = n$, called the *cotangent sheaf*. Writing the (real) basis $\{z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}\}$ for $\mathcal{O}_X(U)$ (local coordinates), we have a formal canonical basis $\{\mathrm{d}z_1, \ldots, \mathrm{d}z_n, \mathrm{d}\overline{z_1}, \ldots, \mathrm{d}\overline{z_n}\}$ for $\Omega_X^1(U)$, hence a canonical decomposition

$$\Omega_X^1(U) = \Omega_X^{1,0}(U) \oplus \Omega_X^{0,1}(U), \qquad \text{with } \Omega_X^{0,1}(U) = \overline{\Omega_X^{1,0}(U)},$$

and

$$I : \mathrm{d}z_i \mapsto \mathrm{d}\overline{z_i},\ \mathrm{d}\overline{z_i} \mapsto -\mathrm{d}z_i$$

is an almost complex structure on $\Omega_X^1(U)$. The *tangent sheaf* of X, denoted by $\mathcal{T}_X$, is its dual $\operatorname{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$, and we have an isomorphism

$$T_x^\vee X = \mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \Omega_{\mathcal{O}_{X,x}/\mathbb{C}} \otimes_{\mathcal{O}_{X,x}} \mathbb{C},$$

hence

$$\mathcal{T}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \xrightarrow{\sim} T_x X,$$

justifying the name "tangent sheaf".

Define a family of sheaves of $\mathcal{O}_X$-modules

$$\Omega_X^k \coloneqq \bigwedge^k \Omega_X^1,$$

in the same way we define the exterior algebra of a vector space. A *differential k-form* on U is then an element $\omega \in \Omega_X^k(U)$, and hence can be written as as a sum

$$\omega = \sum h_{i_1\cdots i_p j_1\cdots j_q} \mathrm{d}z_{i_1} \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z_{j_1}} \wedge \cdots \wedge \mathrm{d}\overline{z_{j_q}}, \qquad p+q=k,\ 1 \le \begin{Bmatrix} i_1 < \ldots < i_p \\ j_1 < \ldots < j_q \end{Bmatrix} \le n,$$

in particular we again have a decomposition

$$\Omega_X^k(U) = \bigoplus_{p+q=k} \Omega_X^{p,q}(U), \qquad \text{where } \Omega_X^{p,q}(U) = \Omega_X^{1,0}(U)^{\wedge p} \wedge \Omega_X^{0,1}(U)^{\wedge q},$$

and elements of $\Omega_X^{p,q}(U)$ are called *(p,q)-forms.*

Define the *exterior differential* to be the unique map $d : \Omega_X^k(U) \to \Omega_X^{k+1}(U)$ such that

(1) d is $\mathbb{C}$-linear,
(2) $d : \Omega_X^0(U) = \mathcal{O}_X(U) \to \Omega_X^1(U) = \Omega_{\mathcal{O}_X(U)/\mathbb{C}}(U)$ coincides with the universal derivation in the definition of the module of relative differential forms,
(3) we have the Leibniz rule, that is, for $\alpha \in \Omega_X^r(U)$ and $\beta \in \Omega_X^s(U)$,

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^r \alpha \wedge d(\beta),$$

(4) $d \circ d = 0$.

We then by construction have the *de Rham complex*

$$\cdots \longrightarrow \Omega_X^{k-1}(U) \xrightarrow{d} \Omega_X^k(U) \xrightarrow{d} \Omega_X^{k+1}(U) \longrightarrow \cdots$$

and the *de Rham cohomology*

$$H_{\mathrm{dR}}^k(U,\mathbb{C}) \coloneqq \frac{\ker(d : \Omega_X^k(U) \to \Omega_X^{k+1}(U))}{\operatorname{im}(d : \Omega_X^{k-1}(U) \to \Omega_X^k(U))}.$$

Elements of the kernel above are *closed k-forms* and those of the image are *exact k-forms.*

A.5. **Kähler manifolds and Chern classes.** A *hermitian metric* on a complex manifold X of (complex) dimension n is a $(1,1)$-form $\omega \in \Omega_X^{1,1}(X)$ such that

(1) $\omega = \overline{\omega}$, and
(2) we can write

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k}$$

such that $h_{jk} = \overline{h_{kj}}$ and the matrix (h_{jk}) is positive definite.

A *hermitian manifold* is a complex manifold that admits a hermitian metric. A hermitian metric ω is *Kähler* if $d\omega = 0$, that is, ω is closed. A *Kähler manifold* is a complex manifold that admits a Kähler metric. The *Kähler class* $[\omega]$ of a Kähler metric ω is its image in $H_{\mathrm{dR}}^2(X,\mathbb{C})$.

Let X be a complex manifold and consider the subsheaf $\mathcal{O}_X^*$ of abelian groups, that is,

$$\mathcal{O}_X^*(U) \coloneqq \mathcal{O}(U)^\times.$$

We have a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

with an induced long exact sequence of sheaf cohomologies

$$\cdots \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \longrightarrow \cdots$$

where $H^1(X,\mathcal{O}_X^*) = \operatorname{Pic} X$ (this is true for all locally ringed spaces X). Then define the *first Chern class* $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ to be the image of its class in $H^1(X,\mathcal{O}_X^*)$ under the connecting map δ above in $H^2(X,\mathbb{Z})$.

Now let X be a smooth projective complex variety and X^{an} its analytification, which is a complex manifold. It turns out that there is a natural Kähler metric on X^{an}, called the *Fubini–Study metric* and denoted by ω_{FS}, and for us we need to know

$$\begin{array}{ccccccc} \operatorname{Pic} X & \longrightarrow & H^2(X,\mathbb{Z}) & \hookrightarrow & H^2(X,\mathbb{C}) & \xrightarrow{\sim} & H^2_{\mathrm{dR}}(X^{\mathrm{an}},\mathbb{C}) \\ \mathcal{O}_X(1) & \longmapsto & c_1(\mathcal{O}_X(1)) & & \longmapsto & & [\omega_{\mathrm{FS}}] \end{array}$$

and in particular every smooth projective complex variety can be considered as a Kähler manifold.

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