Yves André

G-functions and geometry

Typesetter's note

The following is the LaTeXitification of

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We mostly respect the original typography, with exception to (including but not limited to) unifying British spellings, capitalising practices (e.g. Abelian vs. abelian), and situations which necessitate displaying inline equations. We use an entirely different numbering of lemmas, propositions, theorems, corollaries and so on, simply letting `amsthm` do the counting, except in Introduction (therefore, the paragraph titled **References** in Notations is obsolete). We also remove fractions in the numbering of references and similarly simply let `BibLaTeX` count, assuming they were added later to avoid having to globally change other numberings, and we add bibliographic information in the references to help the reader navigate to the source.

We claim no originality except errors. For possible errors in the original text, see errata below.

January 2026 Jiewei Xiong

Possible errata

Consider the work of God:
for who can make that straight, which he hath made crooked?

Ecclesiastes 7:13

Notation: "p. x l. y" means page x, line y from the top (not counting headers; counting equations as one line) and "p. x l. -y" means y lines up from the bottom of page x (counting lines in footnotes). We exclude trivial errors like inconsistent numbering (of e.g. remarks). Add the word "probably" to each of the following sentences.

p. 1 l. 9	$a_0 \cdots a_n$ should be $a_0, \ldots, a_n$
p. 1 l. -10	no such verb form as *lefting*, should be *leaving*
p. 8 l. -11	missing a right bracket after $\overline{\mathbb{Q}}((x))$
p. 9 l. 12	there's no n in statement; instead of $\frac{\mathrm{d}}{\mathrm{d}x}$ it should be $\frac{\mathrm{d}^n}{\mathrm{d}x^n}$, but of course this would be implied by current statement (via induction)
p. 12 l. 16	μ should be ν in denominator
p. 12 l. -4	there's an extra "d" in denominator of first line of equation before x_ν
p. 14 l. 13	X should be W; this is confirmed by l. -6 below when it says "W... ha[s] the same meaning as in Remark 1"
p. 16 l. -8	the final index of the second sum in denominator should be ν; v doesn't show up on left hand side

Foreword

This is an introduction to some geometric aspects of G-function theory. Most of the results presented here appear in print for the first time; hence this text is something intermediate between a standard monograph and a research article; it is not a complete survey of the topic.

Except for geometric chapters (I.3.3, II, IX, X), I have tried to keep it reasonably self-contained; for instance, the second part may be used as an introduction to p-adic analysis, starting from a few basic facts which are recalled in IV.1.1. I have included about forty exercises, most of them giving some complements to the main text.

Acknowledgements

This book was written during a stay at the *Max-Planck-Institut* in Bonn. I should like here to express my special gratitude to this institute and its director, *F. Hirzebruch*, for their generous hospitality. *G. Wüstholz* has suggested the whole project and made its realisation possible, and this book would not exist without his help; I thank him heartily. I also thank *D. Bertrand*, *E. Bombieri*, *K. Diederich*, and *S. Lang* for their encouragements, and *D. Bertrand*, *G. Christol* and *H. Esnault* for stimulating conversations and their help in removing some inaccuracies after a careful reading of parts of the text (any remaining error is however my sole responsibility).

It is a pleasure to acknowledge the influence of previous work of *Bombieri*, *Christol*, and *G. V. Chudnovsky* on this book. Finally, I wish to thank Miss *Grau* for her patience in deciphering and typing the whole manuscript.

May 1988 *Yves André*

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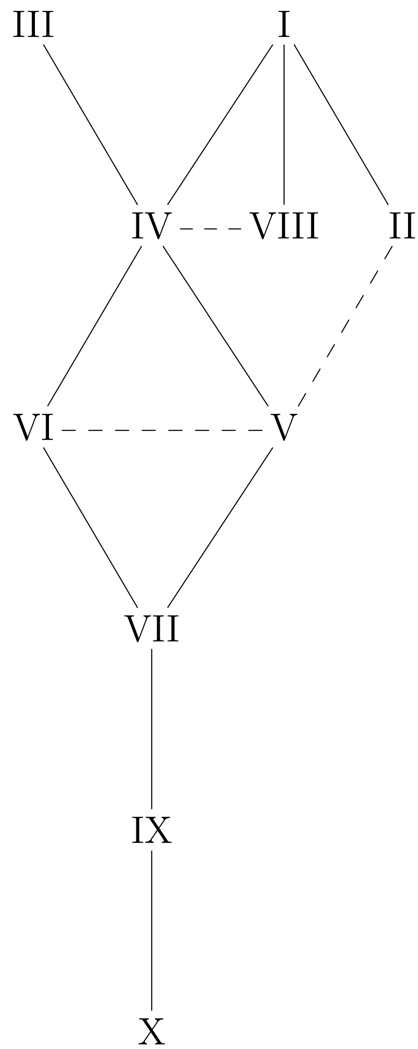
Logical dependence of the chapters



Notations

General notations

$\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ (resp. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$) is the ring (resp. the field) of integers (resp. of rational numbers, of real numbers, of complex numbers). If p is a prime number, $\mathbb{F}_p$ denotes the prime field $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) the ring of p-adic integers (resp. the field of p-adic rational numbers). For $t \in \mathbb{R}$, we shall write $\log^+ t$ for $\log\max(1,t)$; one has $\log^+ t_1t_2 \leq \log^+ t_1 + \log^+ t_2$. We denote by $[t]$ the integral part of t: $[t] \in \mathbb{Z}$, $[t] \leq t < [t]+1$. We denote by $\overline{\lim}$ (resp. $\underline{\lim}$) the upper (resp. lower) limit of a sequence of real numbers. If f, g are two functions of a real variable, with $g \geq 0$, we write $f = O(g)$ if there exists a constant $C > 0$ such that $|f(x)| \leq Cg(x)$ for all sufficiently large x; we write $f = o(g)$ (resp. $f \sim g$) if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ (resp. 1).

Places

Symbols.

$\overline{\mathbb{Q}}$	a fixed algebraic closure of the field of rational numbers,
K	a number field; that is to say, a subfield of $\overline{\mathbb{Q}}$ which is a finite extension of $\mathbb{Q}$,
$\mathcal{O}_K$	the ring of integers in K,
$d = [K : \mathbb{Q}]$	the degree of K over $\mathbb{Q}$,
Σ or $\Sigma(K)$	the set of all places of K,
Σ_f (resp. Σ_∞)	the subset of finite (resp. infinite) places,
$v \mid \mathfrak{p}$ or $\mathfrak{p} = \mathfrak{p}(v)$	v lies above the place $\mathfrak{p}$ of $\mathbb{Q}$,
K_v	a completion of K with respect to $v \in \Sigma$,
$d_v = [K_v : \mathbb{Q}_{\mathfrak{p}(v)}]$	the local degree at $v \in \Sigma$; one has $d = \sum_{v\mid\mathfrak{p}} d_v$.

Normalisation.

$\mid\ \mid_v$	the absolute value in K_v normalised in the following way:
$\quad \lvert\mathfrak{p}(v)\rvert_v = \mathfrak{p}(v)^{\frac{-d_v}{d}}$	if $v \in \Sigma_f$ (ultrametric case),
$\quad \lvert\xi\rvert_v = \lvert\xi\rvert^{\frac{d_v}{d}}$	if $v \in \Sigma_\infty$ (archimedean case), where
$\mid\ \mid$	denotes the euclidean absolute value on K_v, for $v \in \Sigma_\infty$,
$\mathbb{C}_v$	a completion of an algebraic closure of K_v; $\mid\ \mid_v$ extends to $\mathbb{C}_v$,
$i_v : K \hookrightarrow \mathbb{C}_v$ or K_v	the natural embedding.

Remarks. The symbol $\sum_v$ will denote a summation with all $v \in \Sigma(K)$. For any finite extension K' of K, any $\zeta \in K$ and $v \in \Sigma(K)$, one has $|\xi|_v = \prod_{w\in\Sigma(K')} |\zeta|_{K',w}$, and all factors have the same value; see [13],[16] for this material.

Rings

Let R be a commutative entire ring with unit. We shall use the following entire rings (with standard operations):

$R[x]$	the polynomial ring over R; more generally,
$R[\underline{x}]$	the polynomial ring in several commuting indeterminates $\underline{x} = (x_1, \ldots, x_\nu)$ over R,
$R(x)$	the fraction field of $R[x]$,
$R[\![x]\!]$	the ring of formal powers series over R,
$R(\!(x)\!)$	the fraction field of $R[\![x]\!]$,
$M_\mu(R)$	the ring of square matrices of size μ over R; we shall identify $M_\mu(R(\!(x)\!))$ with $M_\mu(R)(\!(x)\!)$,
$\mathrm{GL}_\mu(R)$	the group of its invertible elements,
I or I_μ	its unit,
$\binom{Y}{n}$	for $Y \in M_\mu(R)$, $\binom{Y}{n} = (n!)^{-1}Y(Y-I)\cdots(Y-(n-1)I)$ whenever $n!$ is invertible in R,
${}^tY$	the transposed matrix of $Y \in M_\mu(R)$.

We shall also denote by $M_{\mu,\nu}(R)$ the abelian group of matrices with μ rows, ν columns, whose entries belong to R. For $Y \in M_{\mu,\nu}(R)$, we shall denote by ${}_{ij}Y \in R$ the (i,j)-entry of Y. Let us assume that R is a field. For $Y \in M_{\mu,\nu}(R(\!(x)\!))$, we shall denote by $Y_n \in M_{\mu,\nu}(R)$ the coefficient of x^n in Y, and by ${}_{ij}Y_n \in R$ the coefficient of x^n in ${}_{ij}Y \in R(\!(x)\!)$. For $Y, Z \in M_{\mu,\nu}(R(\!(x)\!))$, the Hadamard product $Y * Z \in M_{\mu,\nu}(R(\!(x)\!))$ is defined by ${}_{ij}(Y * Z)_n = {}_{ij}Y_n \cdot {}_{ij}Z_n$. Then $(M_{\mu,\nu}(R(\!(x)\!)), +, *)$ is a (nonentire) ring with unit; the entries of its unit are $\frac{1}{1-x} \in R(\!(x)\!)$.

Differential operators

Differential polynomials in $\partial = x\frac{\mathrm{d}}{\mathrm{d}x}$ (resp. in $\frac{\mathrm{d}}{\mathrm{d}x}$) and their coefficients, are denoted by Roman (resp. Greek) letters, e.g. $\Lambda = \frac{1}{\mu!}\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} - \sum_{j=0}^{\mu-1} Y_j \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}x^j}$.

References

Quotations like “cf. III(8)”, or “Theorem IV 5.3” indicate a reference to Formula (8) in Chapter III, resp. to the theorem proved in Subsection 5.3 of Chapter IV. When there are several propositions etc... in a single subsection, they are numbered.

Introduction

This booklet is by itself an introduction, because it is the first one devoted to G-function theory. However this does not mean that G-functions constitute a new topic: they were brought in by C. L. Siegel in 1929, in his famous paper on applications of diophantine approximation. He defined G-functions to be the formal power series $y = \sum a_n x^n$ whose coefficients a_n lie in some algebraic number field K, which fulfil the following three conditions:

i) the maximum of the moduli of the conjugates of a_n grows at most geometrically with n (i.e. is bounded by C^n),

ii) there exists a sequence of natural numbers (d_n) which grows at most geometrically such that $d_n a_m$ is integral for every $m \leq n$ (i.e. the "common denominator" of $a_0 \cdots a_n$ grows at most geometrically with n),

iii) y satisfies some linear homogeneous differential equation

$$\frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} y + \gamma_{\mu-1} \frac{\mathrm{d}^{\mu-1}}{\mathrm{d}x^{\mu-1}} y + \cdots + \gamma_0 y = 0$$

with rational function coefficients $\gamma_h \in K(x)$.

After giving some examples, (hypergeometric series ${}_2F_1$, abelian integrals...), Siegel stated some results that he could obtain using the techniques he worked out for so-called E-functions in the same paper, but did not give any detail concerning the proof.

Except for scattered results about particular cases, it was not before forty years later that G-function theory started to develop slowly as a modest chapter of diophantine approximation, in the direction indicated by Siegel. In 1981 a fundamental paper of E. Bombieri appeared, in which not only he proved some of Siegel's irrationality statements in general form (relying on some previous work of A. I. Galočkin), but also, and more significantly, he pointed out the local-to-global nature of the theory.

Since then, the theory overflowed out of its original setting, and new connections with arithmetic algebraic geometry appeared (through the works of D. V. and G. V. Chudnovsky, F. Beukers, and the author); a few of them constitute the matter of the present book.

G-functions and differential equations

Meanwhile, point iii) tended to disappear in the definition of G-functions – maybe because many authors studied components of solutions of linear systems, for which Siegel's definition seems (unduly) insufficient? However this is unfortunate: for instance the (uncountably many) series which satisfy i) and ii) may be quite "pathological", while the (countable) set of G-functions enjoys nice properties, such as the following one (see Chapter VI):

Theorem A. Any G-function $y \in K[\![x]\!]$ satisfies $\prod R_v(y) > 0$, where v runs over the places of K such that the radius of convergence $R_v(y)$ of y (considered as a v-adic function) is finite.

Roughly speaking, this means that the v-adic radii of convergence cannot be too small; whether the converse statement holds, under iii), is an interesting open problem (see Chapter V for a partial answer).

In fact, lefting iii) aside in the definition of G-function is more unfortunate, because it sacrifices the *geometric* nature of Siegel's concept, in light of the following conjecture:

Conjecture. G-functions are exactly the solutions in $\overline{\mathbb{Q}}[\![x]\!]$ of geometric differential equations (over $\overline{\mathbb{Q}}$).

Such a statement is currently believed in by the experts, and our only originality at this point consists in providing a minimal definition of "geometric" differential equations (or polynomials):

Namely they are elements of the multiplicative submonoid of the Weyl algebra $\overline{\mathbb{Q}}\left[x, \frac{\mathrm{d}}{\mathrm{d}x}\right]$ generated by all factors of *Picard–Fuchs* differential polynomials which control the cohomology of smooth varieties over $\overline{\mathbb{Q}}(x)$ (one can even consider only proper smooth varieties without changing the submonoid, see Chapter II).

One of the main aims in the second part of this book is to prove half of this conjecture, namely:

Theorem B. Any solution in $\overline{\mathbb{Q}}[\![x]\!]$ of a geometric differential equation is a G-function.
(See V app.. The difficult case is when 0 is a singularity).

The converse statement seems for the moment to lie beyond the scope of current methods, though some approach already exists via "diagonals" (see Chapter I).

We content ourselves with proving that differential equations satisfied by G-functions share with geometric differential equations very nice p-adic features.

More precisely, let Λ be a differential equation as in point iii) above, and let v be a finite place of K; then we denote by $R_v(\Lambda)$ the supremum of the real numbers $r \leq 1$ such that Λ admits a full set of solutions, analytic in the v-adic disk of radius r, centred at a "generic" point (see Chapter IV): for instance, every geometric differential equation has $R_v = 1$ for almost every v (see Chapter V, Appendix.) We prove in Chapters IV,V,VI the following result:

Theorem C. Let Λ be a differential equation of minimal order, satisfied by a series $y \in K[\![x]\!]$. The following assertions are equivalent:

1) y is a G-function,

2) $\prod_v R_v(\Lambda) > 0$.

The second condition defines what Bombieri calls "Fuchsian differential operator of arithmetic type"; however, for reasons explained in Chapter IV, we shall prefer "G-operator".

Hence we have reduced the above conjecture to a classieal conjecture of Bombieri–Dwork, which asserts that G-operators should be "geometric".

The proof of these theorems combines local methods (weak Frobenius structure...) and global methods (Hermite–Padé approximants, à la Chudnovsky). In fact, we give quantitative results which relate Bombieri's size of y to $\prod R_v(y)$ and $\prod R_v(\Lambda)$. In the same direction, we also compare the algebraic structure of the two sets of functions that the conjecture would identify:

Theorem D. G-functions (resp. solutions of geometric differential equations) form a subspace of $\overline{\mathbb{Q}}[\![x]\!]$ which is stable under both usual (= Cauchy) and coefficientwise (= Hadamard) product.

This includes the following fact: if $\sum y_n x^n$ satisfies some geometric differential equation, so does $\sum y_n^N x^n$ for any positive integer N, whose proof relies heavily upon Hodge theory (degeneration of Leray spectral sequence and semisimplicity of the monodromy for proper smooth morphisms, see Chapter II).

On the other side, the units in the algebra of all G-functions (under the usual product) are exactly the invertible algebraic functions in $\overline{\mathbb{Q}}[\![x]\!]$. Generalising a conjecture of Christol, we expect in addition that G-functions whose inverse satisfies condition ii) above (about denominators) are exactly the diagonals of rational (or algebraic, which amounts to the same) functions.

Special values of G-functions

Via Theorem B, G-functions become a new tool in arithmetic algebraic geometry thanks to the diophantine theory of their "special values", see Chapter VII. The basic result tells that, given G-functions $y_1, \ldots, y_\mu$ and a positive integer δ, there exists a constant c ($\leq$ power of $\delta + 1$) with the following property: for any nonzero integers a, b such that $|b| \geq c|a|^c$, then any polynomial relation $p\left(y_1\left(\frac{a}{b}\right), \ldots, y_\mu\left(\frac{a}{b}\right)\right) = 0$ of degree δ, with coefficients in the base field K, enters as a factor in the specialisation at $x = \frac{a}{b}$ of some functional relation $q(y_1, \ldots, y_\mu) = 0$ between the y_i's (with coefficients in $K(x)$). In fact, this statement has a many-coloured meaning: indeed, one may understand the symbol $y_i\left(\frac{a}{b}\right)$ as the value in the completion K_v taken by the v-adic Taylor series y_i at the point $\frac{a}{b} \in \mathbb{Q} \subset K_v$, for any place v of K such that y_i converges at that point; the constant c does not depend on v.

Bombieri has discovered the possibility of handling several, or even all of these places simultaneously, which leads to a sort of "Hasse principle" for values of G-functions. Using Theorems A and C in order to simplify his hypotheses, one may express this Hasse principle as follows, via the notion of a global relation. According to Bombieri, we say that a relation $p(y_1(\xi), \ldots, y_\mu(\xi)) = 0$ is a global (resp. trivial) relation if it holds v-adically for every place v of K for which $|\xi|_v < \min\left(R_v(y_1), \ldots, R_v(y_\mu), 1\right)$ (resp. if it comes from a functional relation by specialisation at ξ). Then the following finiteness assertion holds true:

Theorem E. Let $\coprod_\delta$ denote the set of points $\xi \in \overline{\mathbb{Q}}$ where there exists some global nontrivial relation of degree δ at ξ between given G-functions $y_1, \ldots, y_\mu$. Then $\coprod_\delta$ has bounded height (at most a power of $\delta + 1$).

In particular, any subset of $\coprod_\delta$ of bounded degree over $\mathbb{Q}$ is finite. In fact Theorem E is effective; the bound for the height depends only on δ, the size of the y_i's, the order of the differential equations they satisfy, the height and the cardinality of the singular locus of these differential equations. Theorem E or simple experiments show that relations between values of algebraic functions at rational points are "almost never" global. Nevertheless global relations may sometimes be found for some carefully chosen ξ in this special case, and this leads eventually to results of a new kind concerning the diophantine geometry of curves. Let us present here two such results:

Theorem F. Let $y^m = q(z)$ define an irreducible curve C, with $q \in \mathbb{Z}[z]$, monic of degree n. Assume moreover that m and n have a prime common factor $\ell \geq 3$. Then

i) there are only finitely many rational points (y, z) on C such that no prime $\equiv 1 \bmod \ell$ divides the denominator of z; in fact, one has the bound $H(z) < 10^{10n^2} H(q)^{8n}$ for any such point (for any polynomial $p \in \mathbb{Q}[z]$, we denote by $H(p)$ the maximum among the absolute values of the numerators and denominators of the coefficients);

ii) there are only finitely many totally real points in $C(\overline{\mathbb{Q}})$ with bounded denominator and degree.

The method of proof of Theorem E is a transcendence argument, namely the so-called Gelfond's method. The same transcendence method, when applied in a different way to series which satisfy properties i) and ii) in the definition of G-functions, furnishes new criteria of rationality. Before giving an example, let us note that $y = \sum a_n x^n$ satisfies i) and ii) iff its size $\sigma(y) := \overline{\lim}_{n\to\infty} \frac{1}{n} h(a_0, \ldots, a_n)$ is finite, where h denotes the logarithmic invariant height on the space K^n.

Theorem G. A series $y \in K[\![x]\!]$ is rational iff for every embedding $K \hookrightarrow \mathbb{C}$, y defines a meromorphic function on a complex disk of radius $> \exp(12\sigma(y))$.

In fact, it is possible to give much stronger variants, assuming for instance only a uniformisation property (cf. Chapter VIII), and this leads to Chudnovsky's criterium of algebraicity, from which they deduce a simple effective proof of the isogeny theorem for elliptic curves over $\mathbb{Q}$.

G-functions and periods of algebraic varieties

Let X be a proper smooth variety over $\overline{\mathbb{Q}} \subset \mathbb{C}$. We call "*period* of X" in degree n, any coefficient divided by $(2i\pi)^n$ of the representative matrix of the canonical isomorphism

$$p_X^n : H^n_{\mathrm{dR}}(X) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^n(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

with respect to bases selected in the algebraic de Rham cohomology $H^n_{\mathrm{dR}}(X) := \mathbb{H}^n(X, \Omega^\bullet_X)$, resp. in the rational singular cohomology of the associated analytic manifold $X^{\mathrm{an}}_{\mathbb{C}}$. An element $t \in H^{2m}_{\mathrm{dR}}(X)$ is called a *Hodge cycle* if it lies at the level F^m of the Hodge filtration, and if $(2i\pi)^{-m} p_X^{2m}(t)$ lies in the rational space $H^{2m}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$. The double rationality feature of Hodge cycles (relatively to the $\overline{\mathbb{Q}}$-space of de Rham cohomology, resp. to the $\mathbb{Q}(2i\pi)$-space of singular cohomology) has the following consequence: every Hodge cycle in $\bigotimes^{2m} H^n_{\mathrm{dR}}(X) \subset H^{2mn}_{\mathrm{dR}}(X^{2n})$ (Künneth) gives rise to polynomial relations with coefficients in $\overline{\mathbb{Q}}(2i\pi)$ among the periods of X (in degree n).

Grothendieck's conjecture. Every polynomial with coefficients in $\overline{\mathbb{Q}}(2i\pi)$ among the periods "comes from Hodge cycles" (see IX.2 for a more precise statement). This is known for elliptic curves with complex multiplication (G. V. Chudnovsky), and for linear relations among periods of any abelian variety (G. Wüstholz). Nevertheless, Grothendieck's conjecture still remains an outstanding open problem in the case of abelian varieties. We present here a new approach via G-functions. Indeed, when X varies in a one-parameter family, the periods are given by the values of analytic functions on the base: the "relative periods", which satisfy suitable Picard–Fuchs differential equations. Moreover, expanding the locally invariant relative periods around a "strong degeneration" (see IX.3,4) in Taylor series, one obtains G-functions – in fact diagonals of rational functions – and it becomes possible to apply the results of the previous paragraph. Making use of results from the theory of variation of Hodge structure, one can prove (IX.5):

Theorem H. Let $X \to S$ be an abelian scheme of relative dimension g over an affine curve S defined over a finite extension K of $\mathbb{Q}$ in $\mathbb{C}$, and let us assume that the fibre of the connected Néron model at some point $s_0 \in (\overline{S}\backslash S)(K)$ is a torus. Let $\delta \geq 0$, and let $s \in S(K)$ be sufficiently close to s_0 in $\overline{S}(\mathbb{C})$ (this proximity condition depends on δ, K, "the" height of s, ... see IX.5). Then every polynomial relation with coefficients in K, of degree $\leq \delta$, between the values at s of the $2g^2$ locally invariant relative periods around s_0, comes from Hodge cycles. In Chapter IX we shall also develop similar results for some projective morphisms more general than abelian schemes; they apparently fall beyond the range of any other current method.

Global relations among periods

We have seen with Theorem E how the existence of global relations leads to much stronger results. Such a favourable situation is encountered in the presence of “exceptional” Hodge cycles in a fibre X_s, for instance when there exist elements of $\operatorname{End} X_s$ which do not come from $\operatorname{End}_S X$. In Chapter X we shall study a typical case, and prove:

Theorem I. Let X/S be an abelian scheme as in Theorem H. Let us assume in addition that the geometric generic fibre is simple of odd dimension g. Then there are only finitely many fibres X_s $(s \in S(\overline{\mathbb{Q}}))$ with bounded residual degree $[K(s) : K]$, for which there is no ring embedding $\operatorname{End} X_s \hookrightarrow M_g(\mathbb{Q})$.

(Note that there does exist an embedding $\operatorname{End}_S X \hookrightarrow M_g(\mathbb{Q})$ because of the degeneration at s_0.)

For $g > 1$ we believe that this type of result is new (and it is “effective”). For $g = 1$, exceptional fibres X_s are elliptic curves with complex multiplication, $[K(s) : K]$ is essentially the class number of the order of complex multiplication, so that the statement is classical: there are only finitely many discriminants with given class number. However our G-function method does not cover this special case – unfortunately, because it would otherwise yield an effective version of Siegel’s theorem which links quantitatively discriminant and class number of definite binary quadratic forms!

Vista: global relations and the “mysterious functor”

The failure of the previous method for $g = 1$ and more generally the need for dealing with all periods of X_s and not only the values at s of the locally invariant relative periods, lead one to expand the relative periods no longer at the degeneration s_0, but instead at some point s_1 of S. This raises at once two problems:

i) the expansions at s_1 are no longer G-functions, but only linear combinations of G-functions, say $y_1, \dots, y_{4g^2}$, with coefficients in the field $K(p_{X_{s_1}})$ generated by the periods of X_{s_1}. The difficulty which arise in constructing special relations among the “archimedean values” $y_1(s), \dots, y_{4g^2}(s)$ (using periods relations on X_s) is often easily overcome by choosing $K(p_{X_{s_1}})$ as small as possible, e.g. X_{s_1} of CM type.

ii) (most serious) How to construct relations between the p-adic values at s of $y_1, \dots, y_{4g^2}$ when s is p-adically close to s_1 and “exceptional” – for instance when $\operatorname{End} X_s$ is bigger than $\operatorname{End}_S X$?

A natural way of dealing with this problem is by imitation of the archimedean case. Here the isomorphism p_{X_s} should be replaced by the functorial $p^{(p)}_{X_s}$ obtained by composing Grothendieck’s “mysterious isomorphism” which relates the de Rham cohomology to the p-adic étale cohomology, and Artin’s isomorphism which links étale and singular cohomology (once a double embedding of the ground number field $K \begin{smallmatrix} \nearrow \mathbb{C} \\ \searrow \mathbb{C}_p \end{smallmatrix}$ is given). J. M. Fontaine and W. Messing (and later G. Faltings in a more general setting) have indeed constructed this “mysterious” isomorphism (which involves the definition of a p-adic analogue of $2i\pi$), and the associated *p-adic periods* (which live in $\mathbb{C}_p((2i\pi))$). By functoriality of $p^{(p)}_{X_s}$, nontrivial endomorphisms on X_s lead to period relations, exactly as in the complex case. Unfortunately, the behaviour of $p^{(p)}_{X_z}$ when X_z varies in a family remains rather mysterious. In fact a solution to the above point ii) seems to depend upon the following:

Problem. How can one relate the p-adic periods of X_s to the values at s of the G-functions $y_1, \dots, y_{4g^2}$? More generally, what are the properties of the mysterious functor with respect to horizontality?

For the applications, the supersingular case is crucial; on the other side, one can raise this problem not only for abelian schemes.

Anyway, a nice answer would be of importance: aside from giving an effective version of Siegel’s theorem as mentioned above, it would also suggest that relations between values at $\xi \in K \subset \mathbb{C}$, say, of solutions $y_1, \dots, y_n$ in $K[\![x]\!]$ of an absolutely irreducible G-operator Λ (for which 0 is ordinary), have a “tendency” to be factors of global relations, thus providing a large range of applications to Theorem E. The heuristic reasons for this are as follows: granting the Bombieri–Dwork conjecture, we may first replace Λ by a Picard–Fuchs equation associated with a proper smooth $K[x]_{(x)}$-scheme X. The Grothendieck conjecture for the product $X_0 \times X_\xi$ would now show that a relation $q(y_1(\xi), \dots, y_n(\xi)) = 0$ with coefficients in K comes from Hodge cycles. Deligne’s hope states that Hodge cycles should be absolute (see Appendix to IX); this would enable us to write similar relations $q_v(y_1(\xi), \dots, y_b(\xi) = 0$ which hold v-adically for every archimedean place v of K for which the v-adic values $y_i(\xi)$ are defined. Furthermore, there is a conjecture of Fontaine which asserts that the “variety of p-adic periods” of X_ξ should be isomorphic to the variety of complex periods (with respect to a double embedding $K \begin{smallmatrix} \nearrow \mathbb{C} \\ \searrow \mathbb{C}_p \end{smallmatrix}$); in fact, this is a consequence of a consequence of a more general conjecture about the behaviour of absolute Hodge cycles under $p^{(p)}_{X_\xi}$. Together with a favourable answer to the absolute problem, this tends to show that there are corresponding relations q_v at the finite places. On multiplying these (finitely many) relations q_v for all v such that $|\xi|_v < R_v(y_1, \dots, y_n)$, we would at last obtain a global elation, "containing" the initial relation as a factor.

For other potential applications, we refer to the last section of Chapter X. In fact, we believe that the above problem is a key for understanding and resolving a whole hierarchy of arithmetico-geometric problems.

The reader may now skip to the last appendix of the book, where a short and typical application of Theorem E is given: a new proof of the transcendence of π.

Here, however, we can only hope that we have given some feeling for the intricate links which relate G-functions to arithmetic algebraic geometry.

Part One

What are G-functions?

Chapter I

G-functions

G-functions appeared in Siegel's paper[17] about diophantine approximation, and led in this context to an extensive literature (see[1] for a small list). In this chapter we present a definition of G-functions (inspired by Bombieri "local-to-global" setting [1]), and define two basic related invariants, namely the size σ (which coincides with Bombieri's one, ibid.) and the global radius. We then turn to examples: rational functions, diagonals, polylogarithms and generalised hypergeometric functions, which we study with some detail; our presentation of diagonals is inspired by Christol [4]. At last we gather some "pathologies".

In the next chapter, we shall explore what *should* be G-functions (conjecturally).

1 Heights and sizes

1.1 Height of algebraic numbers [14]

Let $\zeta \in \overline{\mathbb{Q}}$ an algebraic number, lying in some number field K. If $\zeta \neq 0$, the following "product formula" holds:

$$\sum_{v\in\Sigma(K)} \log|\zeta|_v = 0.$$

The (logarithmic absolute) height of ζ is defined to be

$$\sum_{v\in\Sigma(K)} \log^+|\zeta|_v =: h(\zeta).$$

One has $h(\zeta^r) = |r|h(\zeta)$ for any $\zeta \in K$ and $r \in \mathbb{Q}$. Thanks to our normalisations, $h(\zeta)$ depends only on ζ but not on K. Thus the height is well-defined over $\overline{\mathbb{Q}}$. Let $p = a_0\prod(x-\zeta_i) \in \mathbb{Z}[x]$ the minimal polynomial of ζ over $\mathbb{Z}$. Then the so-called Mahler measure of ζ, defined as $M(\zeta) := |a_0|\prod\max(1,|\zeta_i|)$, is related to the height via the formula

$$\begin{aligned}
[\mathbb{Q}(\zeta):\mathbb{Q}]h(\zeta) &= \log M(\zeta)\\
&= \int_0^1 \log\left|p\left(e^{2\pi t\sqrt{-1}}\right)\right| \,\mathrm{d}t && \text{(Jensen's formula)}\\
&= \overline{\lim_{n\to\infty}}\, \frac{1}{n}\log^+\left|\mathrm{Resultant}\left(p, \sum_{i=0}^n x^i\right)\right| && \text{(Langevin's formula).}
\end{aligned}$$

For a finite family $(A_k)_k$ of matrices, such that all entries belong to K, we set

$$h((A_k)_k) := \sum_{v\in\Sigma(K)} \log^+ \max_{i,j,k} |{}_{ij}A_k|_v.$$

Once again, this quantity does not depend on the choice of the number field which contains the entries ${}_{ij}A_k$ of the A_k's.

The following classical inequality holds:

$$h(AB) \le h(A) + h(B) + \log\nu, \qquad \text{for any } A \in M_{\mu,\nu}(\overline{\mathbb{Q}}),\ B \in M_{\nu,\mu}(\overline{\mathbb{Q}}).$$

On the other hand, given $A \in M_{\mu,\nu}(K)$ of rank $\mu < \nu$, one can find a nonzero matrix $B \in M_{\nu,1}(\mathcal{O}_K)$ such that $AB = 0$, and

$$h(B) \le \frac{\mu}{\nu-\mu}(h(A) + \log\nu + c(K)),$$

where $c(K)$ depends only on K.

Indeed, taking components relative to some $\mathbb{Q}$-basis of K inside $\mathcal{O}_K$, and using the last displayed formula, one sees that it suffices to handle the case $K = \mathbb{Q}$, where it follows easily from the box principle ("Siegel's lemma", which appeared in the same paper [17]); the point is that A carries $\left(\mathbb{Z}_{\substack{\le n \\ \ge -n}}\right)^{\nu}$ into $\left(\mathbb{Z}_{\substack{\le n\nu\|A\| \\ \ge -n\nu\|A\|}}\right)^{\mu}$, so that if $(2n+1)^{\nu} > (2\nu\|A\|n+1)^{\mu}$, then two distinct elements of $\left(\mathbb{Z}_{\substack{\le n \\ \ge -n}}\right)^{\nu}$ have the same image under A, and the difference gives an element of $\left(\mathbb{Z}_{\substack{\le 2n \\ \ge -2n}}\right)^{\nu}$ which is killed by A.

1.2 Height of polynomials

Let $Y \in M_{\mu,\nu}(\overline{\mathbb{Q}}[x]),\ Y = \sum Y_n x^n$. We write as usual $\deg Y = \max\{n : Y_n \neq 0\}$ for $Y \neq 0$. We shall set:

$$h(Y) := (1 + \deg Y)^{-1} h((Y_n)_n).$$

For $\mu = \nu = 1$, it is easy to check that $(1 + \deg y)h(y) \le \sum(h(\zeta) + \log 2)$, where ζ runs over the roots of y.

1.3 Height of formal power series; G-functions

Let $Y \in M_{\mu,\nu}(\overline{\mathbb{Q}}[[x]]),\ Y = \sum_{n \ge c} Y_n x^n$. We denote by $Y_{\le N}$ the truncated series $\sum_{n=0}^{N} Y_n x^n \in M_{\mu,\nu}(\overline{\mathbb{Q}}[x])$. We set:

$$h(Y) := \varlimsup_{N \to \infty} h(Y_{\le N}).$$

This is a well-defined quantity in $[0, \infty]$. One checks immediately that this definition reduces to the previous one when Y has only finitely many (actually $\le 1 + \deg Y$) nonzero coefficients.

Definition 1.3.1. A *G-function* is a formal power series $y \in \overline{\mathbb{Q}}[[x]]$ whose height $h(y)$ is finite, and which is annihilated by some nonzero element of $\overline{\mathbb{Q}}\left[x, \frac{\mathrm{d}}{\mathrm{d}x}\right]$.

Explanation. This is equivalent to the classical definition (Siegel [17]): $y = \sum_{n \ge 0} y_n x^n \in \overline{\mathbb{Q}}[[x]]$ is a G-function if and only if all the coefficients belong to some fixed number field K, and

i) for every $v \in \Sigma_\infty$; $\sum_{n \ge 0} i_v(y_n) x^n \in \mathbb{C}_v[[x]]$ defines an analytic function around 0,

ii) there exists a sequence of natural integers $(d_n)_{n \in \mathbb{N}}$ which grows at most geometrically, such that $d_n y_m \in \mathcal{O}_K$ for $m = 0, \ldots, n$,

iii) y satisfies a linear homogeneous differential equation with coefficients in $K(x)$. This equivalence will be proved in 2.3.

1.4 Size of Laurent series

Let $Y \in M_{\mu,\nu}(\overline{\mathbb{Q}}((x)))\,,\ Y = \sum_{n \ge -N} Y_n x^n$. We set

$$\sigma(Y) := \begin{cases} 0 & \text{if } Y \text{ is a Laurent polynomial (i.e. if almost all coefficients are 0)} \\ h(x^N Y) & \text{otherwise.} \end{cases}$$

One checks immediately that this definition depends only on Y, and not on N. The generalisation to the case of a finite family of matrices is immediate.

We shall also use constantly the convenient notation

$$h_{v,n}(Y) := \frac{1}{n} \max_{\substack{i \le \mu \\ j \le \nu \\ k \le n}} \log^+ |{}_{ij}Y_k|_v\ ;$$

here v denotes a place of some number field K which contains the coefficients ${}_{ij}Y_k$ of the (i,j)-entries of Y for $i \le \mu,\ j \le \nu,\ k \le n$.

However the nonnegative real number $\sum_v h_{v,n}(Y)$ does not depend on the choice of K (by the Remark made in the index of notations).

Lemma 1.4.1. $\sigma(Y) = \varlimsup_{n \to \infty} \sum_v h_{v,n}(Y)$.

Proof. If $Y \in M_{\mu,\nu}(\overline{\mathbb{Q}}\left[x, \frac{1}{x}\right])$, we clearly have $\lim_{n\to\infty} \sum_v h_{v,n}(Y) = 0$, so that it is enough to assume that the sequence $\left(\frac{1}{\varphi(l)}\right)_{l\geq 0}$ of nonzero coefficients of Y is infinite. We then have

$$\sigma(Y) = \overline{\lim_{l\to\infty}} \frac{1}{\varphi(l)} h(Y_0, \ldots, Y_{\varphi(l)}) = \overline{\lim_{l\to\infty}} \frac{1}{\varphi(l)} \sum_v \max_{\substack{i\leq\mu \\ j\leq\nu \\ k\leq\varphi(l)}} \log^+ |{}_{ij}Y_k|_v$$

$$= \overline{\lim_{n\to\infty}} \sum_v \frac{1}{n} \max_{\substack{i\leq\mu \\ j\leq\nu \\ m\leq n}} \log^+ |{}_{ij}Y_n|_v .$$

□

Remark 1.4.2. We could everywhere replace the indexing set of summation $\Sigma(K)$ by Σ_f (resp. Σ_∞). Denoting by h_f, σ_f (resp. h_∞, σ_∞) the corresponding notions – finite (resp. infinite) part of the height or size – the above proof shows that $\sigma_f(Y) = \overline{\lim_{n\to\infty}} \sum_{v\in\Sigma_f} h_{v,n}(Y)$. Assume that all coefficients of the entries of Y lie in a fixed number field K. Let d_n the common denominator in $\mathbb{N}\backslash\{0\}$ of the entries $Y_0, \ldots, Y_n$. One has

$$\sigma_f(Y) \leq \log \overline{\lim_{n\to\infty}} d_n^{\frac{1}{n}} \leq d\sigma_f(Y).$$

The elementary proof is omitted.

Lemma 1.4.3. Let $Y \in M_{\mu,\nu}(\overline{\mathbb{Q}}(\!(x)\!))$.

a) $\max_{i,j} \sigma({}_{ij}Y) \leq \sigma(Y) = \sigma(\zeta Y) \leq \sum_{i,j} \sigma({}_{ij}Y)$, for any $\zeta \in \overline{\mathbb{Q}}$,

b) $\sigma\left(\frac{\mathrm{d}}{\mathrm{d}x} Y\right) \leq \sigma(Y)$, for any $n \in \mathbb{N}$,

c) if the residue Y_{-1} of Y vanishes, $\sigma\left(\int_0^x Y\right) \leq \sigma(Y) + 1$,

d) for $\zeta \in \overline{\mathbb{Q}}$, set $Y_{(\zeta)} := \sum Y_n \zeta^n x^n$. Then $\sigma\left(Y_{(\zeta)}\right) \leq \sigma(Y) + h(\zeta)$.

Let $\left(Y_{[k]}\right)_{k=1}^N$ be a subset of $M_{\mu,\nu}(\overline{\mathbb{Q}}(\!(x)\!))$, then:

e) $\sigma\left(\sum Y_{[k]}\right) \leq \sigma\left(\left(Y_{[k]}\right)_k\right) \leq \sum \sigma\left(Y_{[k]}\right)$,

f) $\sigma\left({}^*Y_{[k]}\right) \leq \sum \sigma\left(Y_{[k]}\right)$,

g) if $\mu = \nu$, $\sigma\left(\prod Y_{[k]}\right) \leq (1 + \log N)\, \sigma\left(\left(Y_{[k]}\right)_k\right)$.

Proof. The proof a,b,d,e,f is straightforward, using Lemma 1.4.1. Let us prove c): by direct computation, we find

$$h_{v,n}\left(\int_0^x Y\right) \leq \begin{cases} h_{v,n}(Y) & \text{if } v \in \Sigma_\infty \\ h_{v,n}(Y) + \dfrac{1}{n} \max_{m\leq n} \log |m|_v^{-1} & \text{if } v \in \Sigma_f, \end{cases}$$

so that $\sigma\left(\int_0^x Y\right) \leq \sigma(Y) + \overline{\lim} \frac{1}{n} \log \operatorname{lcm}(1, 2, \ldots, n)$, and the inequality c) follows from the prime number theorem. In order to prove g), we use a trick introduced in this context by Shidlovsky (see Galočkin [10, Lemma 7]). First we assume without loss of generality that $Y_{[k]} \in M_\mu(\overline{\mathbb{Q}}[\![x]\!])$. Let K be the extension of $\mathbb{Q}$ generated by the m first coefficients ${}_{ij}Y_{kl}$ of the entries ${}_{ij}Y_{[k]}$ of the $Y_{[k]}$'s, and set $Y = \prod_{k=1}^N Y_{[k]}$. We have

$${}_{ij}Y_m = \sum_{\sum m_k = m} \sum_{l_k=1}^{\mu} {}_{il_1}Y_{1m_1\, l_1 l_2}Y_{2_1 m_1} \cdots {}_{l_{N-1}j}Y_{N_1 m_N} .$$

For a finite place $v \in \Sigma_f$, this gives

$$(*) \qquad \log^+ |{}_{ij}Y_m|_v \leq \max_{\substack{m_1+\cdots+m_N=m \\ i_1,\ldots,i_N,\ j_1,\ldots,j_N}} \sum_{k=1}^N \log^+ |{}_{i_k j_k}Y_{k_1 m_k}|_v .$$

By reordering $Y_1, \ldots, Y_N$, we may suppose that $m_1 \geq m_2 \geq \cdots \geq m_N$, hence $km_k \leq m$. This yields

$$\log^+ |{}_{ij}Y_m|_v \leq \sum_{k=1}^N \max_{m_k \leq \frac{m}{k}} \max_{i_k, j_k} \log^+ \left|{}_{i_j, j_k}Y_{k,m_k}\right|_v ,$$

from which we deduce

$$h_{v,m}(Y) \leq \sum_{k=1}^{N} \frac{1}{k} h_{v,\frac{m}{k}}\left(\left(Y_{[l]}\right)_l\right).$$

For an infinite place $v \in \Sigma_\infty$, we have to add an extra term to the right hand side of (∗), namely $\log \#\{m_1, \ldots, m_N : \sum m_k = m\} + \log \mu$, which is $o(m)$; in this case we deduce

$$h_{v,m}(Y) \leq \sum_{k=1}^{N} \frac{1}{k} h_{v,\frac{m}{k}}\left(\left(Y_{[k]}\right)_k\right) + o(1).$$

By summing over $v \in \Sigma(K)$, we find

$$\sigma(Y) \leq \left(\sum_{k=1}^{N} \frac{1}{k}\right) \sigma\left(\left(Y_{[l]}\right)_l\right) \leq (1 + \log N)\, \sigma\left(\left(Y_{[k]}\right)_k\right).$$

□

2 Radii

2.1 Local radii of convergence

Let K be a number field, and let $y = \sum_{n\geq 0} y_n x^n \in K[\![x]\!]$. Then for any $v \in \Sigma_K$, $\sum i_v(y_n)x^n \in \mathbb{C}_v[\![x]\!]$ defines a v-adic Taylor series $y^{(v)}$; we denote by $R_v(y) \in [0, \infty]$ its radius of convergence. By Hadamard's formula, $R_v(y) = \underline{\lim}_{n\to\infty} |y_n|_v^{-\frac{1}{n}}$. More generally, for any Laurent series $y = \sum_{n\geq -N} y_n x^n \in K(\!(x)\!)$, we set $R_v(y) := R_v(x^N y)$; this definition depends only on y but not on N.

2.2 The global radius

For $Y \in M_{\mu,\nu}(K(\!(x)\!))$, we set

$$\rho(Y) := \sum_v \log^+ \left(\min_{i,j} R_v\left({}_{ij}Y\right)\right)^{-1} \in [0, \infty].$$

Lemma 2.2.1. $\rho(Y) = \sum_v \overline{\lim}_{n\to\infty} h_{v,n}(Y)$; ρ is invariant under finite extension of K.

Proof. Hadamard's formula yields

$$\rho(Y) = \sum_v \max_{i,j} \overline{\lim} \frac{1}{n} \log^+ \left|{}_{ij}Y_n\right|_v = \sum_v \overline{\lim} \max_{i,j} \log^+ \left|{}_{ij}Y_n\right|_v.$$

Thus it is enough to show that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \max_{\substack{i,j\\ m\leq n}} \log^+ \left|{}_{ij}Y_m\right|_v = \overline{\lim_{n\to\infty}} \frac{1}{n} \max_{i,j} \log^+ \left|{}_{ij}Y_n\right|_v.$$

This is a special case, for $t_n = \max_{i,j} \log^+ \left|{}_{ij}Y_n\right|_v$, of the well-known inequality

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \max_{m\leq n} t_m \leq \overline{\lim_{n\to\infty}} \frac{t_n}{n} =: \ell.$$

Indeed, for any $\varepsilon > 0$, let $M_\varepsilon \leq N_\varepsilon$ such that $\frac{t_m}{m} \leq \ell + \varepsilon$ for $m \geq M_\varepsilon$ and $\frac{t_m}{m} \leq \frac{N_\varepsilon}{M_\varepsilon}\ell$ for $m < M_\varepsilon$. Then

$$\frac{1}{n} \max_{m\leq n} t_m \leq \max\left(\max_{m\leq M_\varepsilon} \left(\frac{m}{n}\right) \frac{t_m}{m}, \max_{M_\varepsilon\leq m\leq N_\varepsilon} \left(\frac{m}{n}\right) \frac{t_m}{m}\right).$$

The second assertion comes readily from the first one. □

Remark 2.2.2. Here again we could replace the indexing set of summation $\Sigma(K)$ by Σ_f (resp. Σ_∞). The above proof yields corresponding formulae

$$\rho_f(Y) = \sum_{v\in\Sigma_f} \overline{\lim}\, h_{v,n}(Y), \qquad \rho_\infty(Y) = \sum_{v\in\Sigma_\infty} \overline{\lim}\, h_{v,n}(Y).$$

Furthermore $\rho(Y) = \rho_f(Y) = \rho_\infty(Y)$, and $\sigma_\infty(Y) \leq \rho_\infty(Y)$.

Lemma 2.2.3. Let $Y \in M_{\mu,\nu}(K((x)))$.

a) $\max_{i,j} \rho\left({}_{ij}Y\right) = \rho(Y) = \rho(\zeta Y)$, for any $\zeta \in K$,

b) $\rho\left(\frac{\mathrm{d}}{\mathrm{d}x}Y\right) = \rho(Y)$,

c) if the residue Y_{-1} of Y vanishes, $\rho\left(\int_0^x Y\right) = \rho(Y)$,

d) for $\zeta \in K$, $\rho\left(Y_{(\zeta)}\right) \leq \rho(Y) + h(\zeta)$.

Let $\left(Y_{[k]}\right)_{k=1}^N$ be a subset of $M_{\mu,\nu}(K((x)))$, then

e) $\rho\left(\sum Y_{[k]}\right) \leq \rho\left(\left(Y_{[k]}\right)_k\right) = \max_k \rho\left(Y_{[k]}\right)$,

f) $\rho\left({}^*Y_{[k]}\right) \leq \sum \rho\left(Y_{[k]}\right)$,

g) if $\mu = \nu$, $\rho\left(\prod Y_{[k]}\right) \leq \max_k \rho\left(Y_{[k]}\right)$.

Proof. Straightforward. □

2.3

We now prove the equivalence stated in 1.3. Let $y \in K[\![x]\!]$. Assume that $h(y) < \infty$. By Lemmas 1.4.1 and 2.2.1, one gets $\rho_\infty(y) < \infty$ and $\sigma_f(y) < \infty$. The first (resp. second) inequality implies condition 1.3 i) (resp. 1.3 ii), taking into account Remark 1.4.2. Conversely, assume that for any $v \in \Sigma_\infty$, $R_v(y) > 0$ (condition 1.3 i), and that $\overline{\lim}_{n\to\infty} d_n^{\frac{1}{n}} < \infty$ (condition 1.3 ii), where d_n denotes the common denominator in $\mathbb{N}\backslash\{0\}$ of $y_0, \ldots, y_n$. Then

$$\sigma(y) \leq \sigma_\infty(y) + \sigma_f(y) \leq \rho_\infty(y) + \log \overline{\lim_{n\to\infty}}\, d_n^{\frac{1}{n}} < \infty.$$

At last, let $\Lambda = \frac{\mathrm{d}^\mu}{\mathrm{d}x^\mu} - \sum_{j=0}^{\mu-1} Y_j \frac{\mathrm{d}^j}{\mathrm{d}x^j}$; then $\Lambda\left(\sum y_n x^n\right) = 0 \implies \mathbb{Q}(y_0, y_1, \ldots)$ is a number field.

3 Several variables; diagonalisation

3.1

All what precedes extends in a straightforward manner to the case of elements of $K((\underline{x})) = K((x_1, \ldots, x_\nu))$.

For a multi-index $\underline{n} \in \mathbb{N}^\nu$, we denote by $|\underline{n}|$ its length $\sum n_i$; $\underline{x}^{\underline{n}}$ means $\prod x_i^{n_i}$. Let $y = \sum_{\underline{n}} y_{\underline{n}} \underline{x}^{\underline{n}} \in K((\underline{x}))$; for any place v of K, we set

$$h_{v,n}(y) = \frac{1}{n} \max_{|\underline{k}| \leq n} \log^+ |y_{\underline{k}}|_v.$$

We also define the global radius (resp. size) by:

$$\rho(y) = \sum_v \overline{\lim_{n\to\infty}}\, h_{v,n}(y),$$

$$\sigma(y) = \overline{\lim_{n\to\infty}} \sum_v h_{v,n}(y).$$

For $\nu = 1$, previous lemma show the compatibility with original definitions.

3.2 Diagonalisation

One defines the diagonalisation map Δ_ν from $K((\underline{x}))$ to $K((x))$ by the formula

$$\Delta_\nu\left(\sum y_{\underline{n}} \underline{x}^{\underline{n}}\right) = \sum_{n \geq 0} y_{(n,n,\ldots,n)} x^n.$$

This is a useful tool to produce G-functions, through the following lemma (see 4.2):

Lemma 3.2.1. The following inequalities hold:

$$\rho\left(\Delta_\nu(y)\right) \leq \nu\rho(y),$$
$$\sigma\left(\Delta_\nu(y)\right) \leq \nu\sigma(y).$$

Proof. This follows immediately from the obvious inequality

$$h_{v,n}\left(\Delta_\nu(y)\right) \le h_{v,n\nu}(y).$$

□

Remark 3.2.2 (Deligne)**.** Assume that for some infinite place v of K, $y^{(v)} := \sum i_v\left(y_{\underline{n}}\right)\underline{x}^{\underline{n}}$ is analytic at $\underline{0} \in \mathbb{C}_v^\nu$, with $\nu > 1$. Then $\Delta_\nu Y$ is represented by the integral formula

$$\left(2\pi\sqrt{-1}\right)^{-(\nu-1)} = \int_{\substack{|x_2|=\cdots=|x_\nu|=\varepsilon\\ x_1x_2\cdots x_\nu=x}} y\frac{\mathrm{d}x_2\cdots\mathrm{d}x_\nu}{x_2\cdots x_\nu} \qquad \text{for } \varepsilon \text{ and } |x| \text{ small enough.}$$

This follows from the residue formula:

$$\left(2\pi\sqrt{-1}\right)^{-(\nu-1)} = \int_{\substack{|x_2|=\cdots=|x_\nu|=\varepsilon\\ x_1x_2\cdots x_\nu=x}} \underline{x}^{\underline{n}}\frac{\mathrm{d}x_2\cdots\mathrm{d}x_\nu}{x_2\cdots x_\nu} = \begin{cases} x^{n_1} & \text{if } n_1 = n_2 = \cdots = n_\nu\\ 0 & \text{otherwise.}\end{cases}$$

Remark 3.2.3. It seems that diagonals were first introduced in the study of Hadamard products (see e.g. [2]). This relationship is given by the formula

$$\Delta_\nu\left(y_1(x_1),\ldots,y_\nu(x_\nu)\right) = y_1 * \cdots * y_\nu.$$

3.3 Geometric interpretation

Let us set $W = \operatorname{Spec} K(x)[\underline{x}]/(x_1x_2\cdots x_\nu - x)$, with $\nu > 1$. Let (E,∇) be a coherent module with integrable connection over some affine open subset U of W, and let θ be some horizontal $K(U)$-linear map from E to $K(\!(\underline{x})\!)$; in other words, $y := \theta(e)$, for some $e \in \Gamma E$, is a solution in $K(\!(\underline{x})\!)$ of an "integrable differential equation".

We consider the $K(x)$-linear map:

$$\Delta_{\nu,\theta} : e \otimes \frac{\mathrm{d}x_2\cdots\mathrm{d}x_\nu}{x_2\cdots x_\mu} \mapsto \Delta_\nu(\theta(e)), \qquad \text{for all local sections } e \text{ of } E.$$

Proposition 3.3.1. The map $\Delta_{\nu,\theta}$ induces a horizontal map from the algebraic de Rham cohomology group $H^{\nu-1}_{\mathrm{dR}}(U,(E,\nabla))$ endowed with Gauss–Manin connection relative to $K(x)$ (see [11]), to $K(\!(x)\!)$ endowed with exterior derivative.

Proof. The smooth scheme U is affine, thus there is an isomorphism

$$H^{\nu-1}_{\mathrm{dR}}(U,(E,\nabla)) \simeq E_U \otimes \Omega^{\nu-1}_{U/K(x)} \Big/ \nabla_{\nu-1}\left(E_U \otimes \Omega^{\nu-2}_{U/K(x)}\right),$$

where the value at $\frac{\mathrm{d}}{\mathrm{d}x}$ of the Gauss–Manin connection acts through $\nabla\left(\frac{\mathrm{d}}{\mathrm{d}(x_1x_2\cdots x_\mu)}\right)$ on E. The statement would follow from Deligne's integral formula if $\theta(e)^{(v)}$ were analytic at $\underline{0}$ for some $v \in \Sigma_\infty$. However this can fail if $\underline{0}$ corresponds to an irregular singularity of (E,∇); thus we shall rather translate a purely algebraic argument from Christol [4]. The relation $\sum \frac{\mathrm{d}x_i}{x_i} = 0$ in $\Omega^1_{W/K(x)}$, together with the formula $\Delta_\nu\left(x_i\frac{\partial\theta(e)}{\partial x_i}\right) = x\frac{\mathrm{d}}{\mathrm{d}x}\Delta_\nu(\theta(e))$, yields

$$\begin{aligned}
&\Delta_{\nu,\theta}\left(\nabla_{\nu-1}\left(e\otimes\frac{\mathrm{d}x_2\cdots\widehat{\mathrm{d}x_i}\cdots\mathrm{d}x_\nu}{x_2\cdots\widehat{x_i}\cdots\mathrm{d}x_\nu}\right)\right)\\
&= \Delta_{\nu,\theta}\left(\left(x_i\nabla\left(\frac{\partial}{\partial x_i}\right)e - x_1\nabla\left(\frac{\partial}{\partial x_1}\right)e\right)\otimes\frac{\mathrm{d}x_2\cdots\mathrm{d}x_i\cdots\mathrm{d}x_\nu}{x_2\cdots x_i\cdots x_\nu}\right)\\
&= \Delta_\nu\left(x_i\frac{\partial\theta(e)}{\partial x_i} - x_1\frac{\partial\theta(e)}{\partial x_1}\right) = 0.
\end{aligned}$$

Therefore $\Delta_{\nu,\theta}$ factors through $H^{\nu-1}_{\mathrm{dR}}(U,(E,\nabla))$. In order to prove the horizontality statement, we fix $x_2,\ldots,x_\nu$ and get

$$\Delta_{\nu,\theta}\left(x_1\nabla\left(\frac{\partial}{\partial x_1}\right)e\otimes\frac{\mathrm{d}x_2\cdots\mathrm{d}x_\nu}{x_2\cdots x_\nu}\right) = \Delta_\nu\left(x_1\frac{\partial\sigma(e)}{\partial x_1}\right) = x\frac{\mathrm{d}}{\mathrm{d}x}\Delta_{\nu,\theta}\left(e\otimes\frac{\mathrm{d}x_2\cdots\mathrm{d}x_\nu}{x_2\cdots x_\nu}\right).$$

□

Corollary 3.3.2. Assume that $H^{\nu-1}_{\mathrm{dR}}(U,(E,\nabla))$ is finite-dimensional over $K(x)$ (assume for instance that (E,∇) has only regular singular points, see next chapter, 2.2) then for $y=\theta(e)$ as above, $\Delta_\nu(y)$ satisfies an ordinary linear homogeneous differential equation with coefficients in $K(x)$.

Corollary 3.3.3. Assume that θ is a solution in $K(\!(\underline{x})\!)$ of the Picard–Fuchs system $H^{\mu}_{\mathrm{dR}}(Y/K(\underline{x}))$ of a smooth proper $K(\underline{x})$-variety Y. Then $\Delta_{\nu,\theta}$ is a solution in $K(\!(x)\!)$ of the Picard–Fuchs system $H^{\mu+\nu-1}_{\mathrm{dR}}(Z/K(x))$ of a smooth $K(x)$-variety Z.

Proof. Let V be an open dense subset of $\operatorname{Spec} K\left[\underline{x},\frac{1}{x_1\cdots x_\nu}\right]$ such that Y extends to a smooth proper morphism $Y_V \xrightarrow{f} V$, and let us denote by g the obvious smooth morphism $V\to \operatorname{Spec} K\left[x_1\cdots x_\nu,\frac{1}{x_1\cdots x_\nu}\right]$. Let us consider the cartesian squares:

$$\begin{array}{ccc} Z & \longrightarrow & Y_V \\ \downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle g} \\ W & \longrightarrow & \operatorname{Spec} K\left[\underline{x},\frac{1}{x_1\cdots x_\nu}\right]. \end{array}$$

According to the proposition, $\Delta_{\nu,\theta}$ is a solution in $K(\!(x)\!)$ of $H^{\nu-1}_{\mathrm{dR}}(U/K(x),H^{\mu}_{\mathrm{dR}}(Z/U))$.

On the other hand, there is the Leray spectral sequence

$$(*)\qquad H^{\nu-1}_{\mathrm{dR}}(U/K(x),H^{\mu}_{\mathrm{dR}}(Z/U)) \Longrightarrow H^{\mu+\nu-1}_{\mathrm{dR}}(Z/K(x)).$$

Let us extend the scalars K to $\mathbb{C}$; since $f_{\mathbb{C}}$ is proper and smooth, the Leray spectral sequence of local systems $R^{\nu-1}g_{\mathbb{C}*}R^{\mu}f_{\mathbb{C}*}(\mathbb{C}) \Rightarrow R^{\mu+\nu-1}\left(gf_{\mathbb{C}}\right)_*(\mathbb{C})$ degenerates [6, 2.4]. It follows from the comparison theorem that $(*)$ also degenerates as a spectral sequence of $K(x)$-vector spaces with connection. Thus $\Delta_{\nu,\theta}$ is a solution of $H^{\mu+\nu-1}_{\mathrm{dR}}(Z/K(x))$. □

Remark 3.3.4. Combining Corollary 3.3.3 with Remark 3.2.3, we get that if $\sum a_n x^n$ satisfies a Picard–Fuchs equation from projective geometry, then for any N, $\sum a_n^N x^n$ satisfies a Picard–Fuchs equation.

4 Examples

We shall study four typical classes of G-functions, each of which is stable under Hadamard product; namely: rational functions, diagonals of rational functions in several variables, polylogarithms and hypergeometric functions (geometric and hypergeometric series were already put forward by C. L. Siegel [17], and G-functions borrow their generic name from these special cases). Each of these series satisfies some linear homogeneous differential equation, which turns out to come from geometry.

4.1 Rational functions

Let $y\in K(x)$, and let us write $\operatorname{pol}(y)$ for the set of poles of y. We may write y as the quotient $\frac{p}{q}$ of two polynomials in $\mathcal{O}_K[x]$. Let us write N for the norm of the first nonzero coefficient of q; then $y\in\mathcal{O}_K\left[\frac{1}{N}\right](\!(x)\!)$. On the other hand, it is immediate that $\rho_\infty(y)<\infty$. Since such series occur frequently, we state a

Definition 4.1.1 (Christol)**.** A Laurent series $y\in K(\!(x)\!)$ is *globally bounded* if and only if

i) for any $v\in\Sigma(K)$, $R_v(y)>0$,

ii) there exists $N\in\mathbb{N}^\times$ such that $y\in\mathcal{O}_K\left[\frac{1}{N}\right](\!(x)\!)$.

Lemma 4.1.2. Any $y\in K(x)$ satisfies $\rho(y)=\sigma(y)=h(\operatorname{pol}(y))$.

Proof. We have $R_v(y)=\min_{\zeta\in\operatorname{pol}(y)}|\zeta|_v$ for any $v\in\Sigma(K)$, whence the equality $\rho(y)=h(\operatorname{pol}(y))$.

On the other side, the fact that y is globally bounded implies $h_{v,n}(y)=0$ for almost all v, and all n. Using Lemmas 1.4.1 and 2.2.1, we come by the inequality $\sigma(y)\le\rho(y)$. In order to show that it is an equality, it suffices to establish the existence of the limit $\lim_{n\to\infty}h_{v,n}(y)$ for any $v\in\Sigma(K)$; but this follows from the fact that the coefficients of y satisfy linear recurrence equations for $n\gg 0$ (see Remark 4.1.4 below), or by decomposition into simple elements. □

Remark 4.1.3. This lemma, together with the identity $\frac{\mathrm{d}}{\mathrm{d}x}\frac{p}{q} = \left(\frac{p'}{p} - \frac{q'}{q}\right)\frac{p}{q}$ show that rational functions are G-functions.

Remark 4.1.4. The lemma generalises immediately to the case of a matrix $Y \in M_{\mu,\nu}(K(x))$. The stability of $M_{\mu,\nu}(K(x))$ under Hadamard product is easily seen using the characterisation of rational series: $y \in K(x) \iff \exists N \in \mathbb{N}^\times,\ \exists Y, Z \in M_N(K)$ such that $Y_n = \operatorname{tr} Y Z^n$ (existence of recurrence relations); we have the formula $(Y_1 * Y_2)_n = \operatorname{tr}(Y_1 \otimes Y_2)(Z_1 \otimes Z_2)^n$, with obvious notations.

4.2 Diagonals of rational functions

We shall denote by $K[\underline{x}]_{(\underline{x})}$ the localisation of the ring $K[\underline{x}] = K[x_1, \dots, x_\nu]$ at the ideal generated by $x_1, \dots, x_\nu$, and by $K\{x\}$ the henselisation of $K[x]$ at the ideal generated by x (i.e. the subring of $K[\![x]\!]$ of algebraic elements over $K(x)$).

Definition 4.2.1. Elements in the target $\Delta_\nu\left(K[\underline{x}]_{(\underline{x})}\right)$ of the diagonalisation map restricted to $K[\underline{x}]_{(\underline{x})}$ are called *diagonals of rational functions* (over K).

Remark 4.2.2. Let us consider again the geometric interpretation of Δ_ν in 3.3. In the present case, let $\frac{p}{q} \in K[\underline{x}]_{(\underline{x})}$, with $p, q \in K[\underline{x}]$. We may take for U the subset of X where q does not vanish; $E = \mathcal{O}_U$, endowed with exterior derivative ∇; θ: the standard horizontal map $\mathcal{O}_U \to K(\!(\underline{x})\!)$, where x is replaced by $x_1 x_2 \cdots x_\nu$; $e := \frac{p}{q}$. We have $H^{\nu-1}_{\mathrm{dR}}(U, (E, \nabla)) = H^{\nu-1}_{\mathrm{dR}}(U)$, the ordinary algebraic de Rham cohomology of the smooth affine scheme U. This is a finite-dimensional $K(x)$-vector space; see [15] for an algebraic proof which does not use resolution of singularities. According to Corollary 3.3.3, diagonals of rational functions satisfy "Picard–Fuchs" differential equations associated to smooth affine $K(x)$-schemes.

Lemma 4.2.3. Let $y \in K[\![x]\!]$, $y = \Delta_\nu\left(\frac{p}{q}\right)$ be a diagonal of rational function. Then y is a globally bounded G-function, and $\sigma(y) \le \rho(y) < \infty$.

Proof. We may assume that $p, q \in \mathcal{O}_K[\underline{x}]$; let us denote by N the norm of $q(0) \neq 0$. Then it is clear that $\frac{p}{q} \in \mathcal{O}_K$ and $y \in \mathcal{O}_K\left[\frac{1}{N}\right][\![x]\!]$. On the other side, the v-adic radius of convergence $R_v\left(\frac{p}{q}\right)$ is nonzero for every $v \in \Sigma(K)$, and the same holds for $R_v(y)$ according to Hadamard's formula. Using the last remark, this shows that y is a globally bounded G-function. The deduction $\sigma(y) \le \rho(y)$ is made as in Lemma 4.1.2. In fact, it could be shown that $\sigma_f(y) = \rho_f(y) \le \nu h_f\left(q(\underline{0})^{-1}\right) \le \nu h\left(q(\underline{0})\right)$. □

It happens that diagonals of rational functions occur very frequently, even though it is often difficult to find the (nonunique) relevant rational function. To explain this fact, G. Christol [5] has set the following conjecture up:

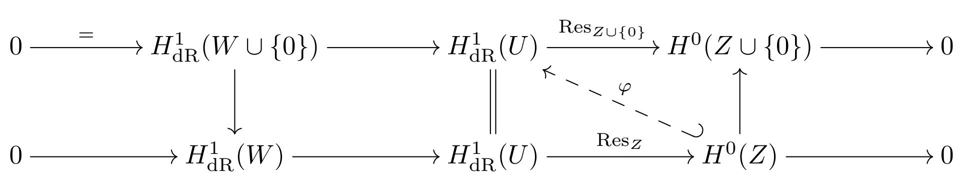
Conjecture 4.2.4. Every globally bounded solution in $K[\![x]\!]$ of a linear homogeneous differential equation with coefficients in $K[x]$ is the diagonal of some rational function.

In other words, every globally bounded G-function should be a diagonal. We now prove that algebraic functions are diagonals of rational functions in two variables (Christol–Furstenberg [3][9]). Consequently, they are globally bounded (Eisenstein).

Proposition 4.2.5. The equality $\Delta_2\left(K[x_1, x_2]_{(x_1,x_2)}\right) = K\{x\}$ holds.

Sketch of proof. In fact we shall only consider the inclusion $\supset$. Let $y \in K\{x\}$ and let $r(y, x) := 0$ be a polynomial equation for y. Assuming that $r(0,0) = 0$, $\left.\frac{\partial r}{\partial y}\right|_{(0,0)} \neq 0$, $\left.\frac{\partial r}{\partial x}\right|_{(0,0)} \neq 0$, we shall exhibit a rational function $\frac{p}{q}$ such that $\Delta_2\left(\frac{p}{q}\right) = y$. We set $q(x_1, x_2) = \frac{1}{x_1} r(x_1, x_1 x_2)$, so that $\frac{1}{q} \in K[x_1, x_2]_{(x_1,x_2)}$, and $\left.\frac{\partial q}{\partial x_2}\right|_{(0,0)} \neq 0$.

Let us consider the following diagram (where W and U have the same meaning as in Remark 4.2.2, and $Z = W \backslash U$):



where all arrows are horizontal maps, and where the horizontal rows are the residue exact sequences: Res_Z is the "coefficient of $\frac{\mathrm{d}q}{q}$", given at the stage of differential forms by

$$\mathrm{Res}_Z\left(\frac{p}{q}\frac{\mathrm{d}x_2}{x_2}\right) = \left(\frac{\partial q}{\partial x_2}\right)^{-1} \left.\frac{p}{x_2}\right|_{q(x_1,x_2)=0}.$$

Now the derivation $\frac{\mathrm{d}}{\mathrm{d}x}$ extends in a unique way to $K(x,y)$, whence a connection on this space, which can be identified with the Gauss–Manin connection on $H^0(Z)$. It follows that the image of $y \in K(x,y) \simeq H^0(Z)$ under φ is given by the class of $\frac{p}{x} \cdot \frac{\mathrm{d}x_2}{x_2}$ where $p = x_1 x_2 \frac{\partial q}{\partial x_2}$.

The following diagram of horizontal maps

$$\begin{array}{ccccc} H^1_{\mathrm{dR}}(U) & \xleftarrow{\varphi} & H^0(Z) & \xleftarrow{\approx} & K(x,y) \\ \Big\downarrow{\scriptstyle \Delta_{2,\theta}} & & & & \Big\downarrow \\ K[\![x]\!] & = & = & = & K[\![x]\!] \end{array}$$

(where θ is defined in the above remark) shows that $(\Delta_{2,\theta} \circ \varphi)\,(Y)$ satisfies the same differential equation as y, and

$$(\Delta_{2,\theta} \circ \varphi)\,(y)|_0 = x\Delta_2 \left(\frac{1}{q} \frac{\partial q}{\partial x_2} \right)\Big|_0 = 0.$$

It follows that $y = \Delta_2 \left(\frac{x_1 x_2}{q} \frac{\partial q}{\partial x_2} \right)$. □

For a proof of the reversed inclusion $\subset$, with an argument from linguistics, see [8, 5].

Remark 4.2.6. The stability of diagonals of rational functions under Hadamard product is immediate from the formula:

$$\Delta_{\nu_1+\nu_2}\left(r_1\left(x_1,\ldots,x_{\nu_1}\right) r_2\left(x_{\nu_1+1},\ldots,x_{\nu_1+\nu_2}\right)\right) = \Delta_{\nu_1} r_1 * \Delta_{\nu_2} r_2.$$

However the subclass of algebraic functions is not stable under $*$; by way of counterexample, one may take (Jungen, 1931):

$$\begin{aligned}(1-x)^{\frac{1}{2}} * (1-x)^{-\frac{1}{2}} &= \Delta_4 \left(\frac{4}{(2-x_1-x_2)(2-x_3-x_4)} \right) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, x\right) \\ &= \sum_{n\geq 0} \binom{2n}{n}^2 \left(\frac{x}{16}\right)^n,\end{aligned}$$

which is transcendental.

4.3 Polylogarithms

We turn back to more down-to-earth examples. Let $L_k = \sum_{n\geq 0} \frac{x^n}{n^k}$ be the kth polylogarithmic series. It satisfies the "unipotent" differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1-x}{x} \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^k L_k = 0$$

obtained from the chain rule $x\frac{\mathrm{d}}{\mathrm{d}x} L_k = L_{k-1},\ L_0 = \frac{x}{1-x}$; the other solutions can be expressed by means of the functions $1, \log x, \ldots, \log^{k-1} x$; the singularities are $0, 1, \infty$.

Lemma 4.3.1. One has $\rho(L_k) = 0,\ \sigma(L_k) = k$.

Proof. This is a straightforward consequence of the prime number theorem. □

Moreover, one can show that $\overline{\lim}_{k\to\infty} \frac{\sigma(L_1^k)}{\log k} = 1$ (cf. Lemma 2.2.3g); see ch. VIII.

Remark 4.3.2. Integration of any formal power series y is nothing but the Hadamard product $xy * L_1$.

4.4 Generalised hypergeometric functions

For $a \in \mathbb{Q}$, we set $(a)_0 = 1,\ (a)_{n+1} = (a+n)(a)_n$, and for $\underline{a} := (a_1, \ldots, a_\mu) \in \mathbb{Q}^\mu$ we set $(\underline{a})_n = \prod_{m=1}^{\mu} (a_m)_n$. To any couple $(\underline{a}, \underline{b})$ in $(\mathbb{Q}\backslash\{-\mathbb{N}\})^\mu \times (\mathbb{Q}\backslash\{-\mathbb{N}\})^\nu$, we associate the hypergeometric function

$$y = F(\underline{a}, \underline{b}, x) := \sum_{n\geq 0} \frac{(\underline{a})_n}{(\underline{b})_n} x^n.$$

Lemma 4.4.1. The three conditions $\rho(y) < \infty$, $\sigma(y) < \infty$ and $\mu = \nu$ are equivalent. If they are satisfied, one has

$$\rho(y) = \sigma(y) \leq \max\left(\sum_{m=1}^{\mu} \left(2h_f(a_m) - hf(b_m)\right), 0 \right).$$

Proof. Either of the conditions $\rho(y) < \infty$, $\sigma(y) < \infty$ implies that for $v \in \Sigma_\infty$, $R_v(y) > 0$ which implies in turn that $\mu \leq \nu$, and $R_v(y) \geq 1$ (hence $\rho_\infty(y) = \sigma_\infty(y) = 0$). Let N be the greatest common denominator of the a_m, b_m's; for $p > N$ and $n \to \infty$, we have

$$\left|\frac{(a_m)_n}{(b_m)_n}\right|_p = O\left(p^{\log n}\right),$$
$$\left|\frac{1}{(b_n)_n}\right|_p^{\frac{1}{n}} \sim p^{\frac{1}{p-1}},$$
$$\operatorname{den}\left(\frac{N^n(a_m)_n}{(b_m)_n}\right) = O\left(e^{\frac{1}{\log n}}\right),$$

and $\left(\operatorname{den}\frac{N^n}{(b_m)_n}\right)^{\frac{1}{n}} \sim \frac{n}{e}$ (Stirling, see the appendix). The former two estimates, together with the divergence of $\sum_{p>N}\frac{\log p}{p-1}$, show that $\rho(y) < \infty \implies \mu \geq \nu$.

The latter two estimates show that $\sigma(y) < \infty \implies \mu \geq \nu$. Conversely the first and the third estimates show that $\mu = \nu$ implies finiteness for ρ and σ, and that

$$\rho(y) = \sum_{p|N} \varlimsup_{n\to\infty} h_{p,n},$$
$$\sigma(y) = \varlimsup_{n\to\infty} \sum_{p|N} h_{p,n}.$$

A straightforward computation (remaking that $|(a_m)_n|_p = |a_m|_p^n$ if $|a|_p > 1$) then leads to the inequality

$$\rho(y) = \sigma(y) \leq \max\left(\sum_{m=1}\left(2\log\operatorname{den} a_m - \log\operatorname{den} b_m\right), 0\right).$$

□

Remark 4.4.2. We could define hypergeometric series for parameters $(\underline{a}, \underline{b})$ in $(K\backslash\{-\mathbb{N}\})^{\mu+\nu}$ for any number field. However it follows from methods of Chapter VI that such a hypergeometric series is a G-function only if $(\underline{a}, \underline{b}) \in (\mathbb{Q}\backslash\{-\mathbb{N}\})^{\mu+\nu}$, see VI ex. 1.

Remark 4.4.3. G. Christol [5] has determined all globally bounded hypergeometric functions. The extra condition is the following one: let N as above; then for any M with $0 \leq M \leq N$ and $(M, N) = 1$, and for any positive integer j with $j \leq \mu$, $\#\{i : Ma_i \prec Mb_j\} \geq \#\{i : Mb_i \prec Mb_j\}$; here $\prec$ is the total ordering of $\mathbb{R}$ defined by

$$y \prec z \iff y + [-y] < z + [-z] \text{ or } (y + [-y] = z + [-z] \text{ and } y \geq z).$$

Let us now introduce the classical Meijer G-functions, which however are *not* G-functions in Siegel's sense! These are integrals of Mellin–Barnes type over a suitable loop:

$$G^{m,n}_{\nu,\mu}(\underline{a}, \underline{b}, x) := \frac{1}{2\pi\sqrt{-1}} \oint \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{m}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{\mu}\Gamma(1 - b_j + s)\prod_{j=n+1}^{v}\Gamma(a_j - s)} x^s \, \mathrm{d}s \qquad \text{for } 0 \leq m \leq \mu,\ 0 \leq n \leq \nu.$$

In the case $\mu = \nu$, these functions satisfy some *Fuchsian* differential equation. Namely,

$$z := G^{m,n}_{\mu,\mu}(\underline{a}, \underline{b}, (-1)^{m+n}x)$$

satisfies the equation

$$(*) \qquad (-1)^\mu x = \prod_{j=1}^{\mu}(\partial - a_j + 1)z = \prod_{j=1}^{\mu}(\partial - b_j)z \qquad \text{where } \partial = x\frac{\mathrm{d}}{\mathrm{d}x},$$

whose singularities are $x = 0, (-1)^\mu$ and ∞.

The link with hypergeometric series is given by the formulae

$$F(\underline{a}, \underline{b}, x) = \frac{\prod_{j=1}^{\mu}\Gamma(b_j)}{\prod_{j=1}^{\mu}\Gamma(a_j)} G^{\mu,1}_{\mu,\mu}\left(\underline{a}, \underline{b}, -\frac{1}{x}\right) = \frac{\prod_{j=1}^{\mu}\Gamma(b_j)}{\prod_{j=1}^{\mu}\Gamma(a_j)} G^{1,\mu}_{\mu,\mu}\left(\underline{1} - \underline{a}, \underline{1} - \underline{b}, x\right)$$

and

$$G^{m,n}_{\mu,\mu}(\underline{a},\underline{b},x) = \sum_{k=1}^{m} \frac{\prod_{\substack{j=1\\ j\neq k}}^{m}\Gamma(b_j-b_k)\prod_{j=1}^{n}\Gamma(1+b_k-a_j)}{\prod_{j=m+1}^{\mu}\Gamma(1+b_k-b_j)\prod_{j=n+1}^{\mu}\Gamma(a_j+b_k)} x^{b_k} F\left(-\underline{a}+\underline{1+b_k}, -\underline{b}+\underline{1+b_k}, (-1)^{\mu-m-n}x\right),$$

where we set $\underline{h} = (h,\dots,h)$ for any $h \in \mathbb{Q}$, see [7, 5.5]. The latter formula shows that $G^{m,n}_{\mu,\mu}$ is a linear combination (with transcendental constant coefficients) of some Siegel G-functions.

Remark 4.4.4. In the case $\mu = \nu = 1$, we have $F(a,b,x) = {}_2F_1(a,1,b,x)$, the classical hypergeometric function, and it is well known that equation $(*)$ is a factor of a Picard–Fuchs equation [12]. For higher μ, this is by no means obvious. However it remains that:

Proposition 4.4.5 (for $\mu = \nu$)**.** $F(\underline{a},\underline{b},x)$ satisfies some Picard–Fuchs differential equation.

Proof. According to remarks of 4.2, we have

$$F(\underline{a},\underline{b},x) = \mathop{*}_{i=1}^{\nu}\left({}_2F_1\left(a_i,1,b_i,x\right)\right) = \Delta_\nu\left(\prod_{i=1}^{\nu}{}_2F_1\left(a_i,1,b_i,x_i\right)\right).$$

By Corollary 3.3.3, it suffices to show that $\prod_{i=1}^{\nu}{}_2F_1\left(a_i,1,b_i,x_i\right)$ satisfies a Picard–Fuchs differential equation associated $H^{\mu'}_{\mathrm{dR}}(Y/\mathbb{Q}(\underline{x}))$ for some proper smooth Y. Using Künneth formula in algebraic de Rham cohomology, it is enough to prove this statement for $\nu = 1$. If $b \in \mathbb{N}^\times$, then ${}_2F_1(a,b,x)$ is algebraic and the statement holds with $\mu^1 = 0$. If $b \notin \mathbb{N}^\times$, we use Gauss relations between contiguous hypergeometric series

$$(b-a-1){}_2F_1(a,1,b,x) + a{}_2F_1(a+1,1,b,x) - (b-1){}_2F_1(a,1,b-1,x) = 0$$
$$b[a-(b-a)x]{}_2F_1(a,1,b,x) + ab(1-x){}_2F_1(a+1,1,b,x) + (b-1)(b-a)x{}_2F_1(a,1,b+1,x) = 0$$

in order to reduce ourselves to the case $a > 0$, $1 > b > 2$. In this case, Euler's integral representation

$${}_2F_1(a,1,b,x) = (b-1)\int_0^1 (1-t)^{b-2}(1-tx)^{-a}\,\mathrm{d}t$$

show that ${}_2F_1(a,1,b,x)$ satisfies the Picard–Fuchs equation associated to the differential $\frac{\mathrm{d}t}{u}$ over the smooth completion of the curve

$$u^N = (1-t)^{(2-b)N}(1-tx)^{aN}, \qquad N = \mathrm{den}(a,b).$$

□

5 Counterexamples

In this paragraph, we gather some "pathological" examples to show that there is no link in general between ρ and σ. We shall show later that for solutions of linear homogeneous differential equations with coefficients in $\overline{\mathbb{Q}}(x)$, ρ and σ are in contrast closely related. We also state that ρ and σ are bad-behaved under inversion of functions.

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Glossary of notations