A GLOSSARY OF ALGEBRAIC GEOMETRY

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Contents

1. Algebra

Definition 1.1 (Properties of a ring)**.** Let A be a ring.

(1) A is *reduced* if for each $a \in A$, we have $a^2 = 0 \implies a = 0$ (that is, A has no nonzero nilpotent elements).
(2) A is *normal* if the localisation $R_\mathfrak{p}$ at each prime ideal $\mathfrak{p}$ of R is a normal (integrally closed) domain, that is,
$$R_\mathfrak{p} = \{s \in \operatorname{Frac}(R_\mathfrak{p}) : \exists \text{ monic } g \in R_\mathfrak{p}[x] : g(s) = 0\}.$$
(3) In the case that A is noetherian, we say A is *regular* if each $R_\mathfrak{p}$ is regular, that is, the Krull dimension of $R_\mathfrak{p}$ equals the minimal number of generators of its maximal ideal.

Definition 1.2 (Properties of a ring map)**.** Let $f : A \to B$ be a ring map.

(1) f is *flat* if the functor $- \otimes_A B : \mathsf{Mod}_A \to \mathsf{Mod}_A$ is exact, that is, $B_1 \to B_2 \to B_3$ is exact implies $B_1 \otimes_A B \to B_2 \otimes_A B \to B_3 \otimes_A B$ is exact for A-modules B_1, B_2, B_3.
(2) f is *faithfully flat* if moreover the converse implication holds.
(3) f is *integral* if for each $b \in B$, there is a monic $g \in A[x]$ such that $\overline{g}(b) = 0$ (there is a slight abuse of notation: the last g denotes the image of the first g in $B[x]$).
(4) f is *finite* if B is a finitely generated A module, that is, there a surjective $A^{\oplus n} \to B$ for some n. Every finite ring map is integral.
(5) In the case that A and B are local with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$, we say f is a *local ring map* if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Definition 1.3. A local ring map $f : A \to B$ with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ is *unramified* if $f(\mathfrak{m}_A)B = \mathfrak{m}_B$, and $B/\mathfrak{m}_B$ is a finite separable extension of $A/\mathfrak{m}_A$.

2. Geometry

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. $(X, \mathcal{O}_X)$ is a *locally ringed space* if each stalk $\mathcal{O}_{X,x}$ is a local ring. In this case we denote the (unique) maximal ideal of $\mathcal{O}_{X,x}$ by $\mathfrak{m}_x$, and the residue field by $\kappa(x)$.

A *morphism of locally ringed spaces* $(\pi, f) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair where $\pi : X \to Y$ is a continuous map of topological spaces and $f : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ is a natural transformation of functors, such that the induced map $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ of stalks is a local ring map.

Definition 2.2. A *scheme* is a ringed space locally isomorphic to affine $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. They are locally ringed spaces. A *morphism of schemes* is a morphism of them as locally ringed spaces.

An S-scheme refers to a scheme X with a fixed morphism $X \to S$, and for a ring A, an A-scheme is a $\operatorname{Spec} A$-scheme.

Definition 2.3. Let $f : X \to Y$ be a morphism of schemes. The *diagonal morphism* of f is the unique morphism $\delta_{X/Y} : X \to X \times_Y X$ such that $\mathrm{pr}_1 \circ \delta_{X/Y} = \mathrm{pr}_2 \circ \delta_{X/Y} = \mathrm{id}_X$.

Definition 2.4. Let $(X, \mathcal{O}_X)$ be a scheme and $U \subset X$ an open subset. Then $(U, \mathcal{O}_{X|U})$ is a scheme, and we say U is *affine open* in X if this $(U, \mathcal{O}_{X|U})$ is affine.

Definition 2.5 (Properties of a scheme)**.** Let $(X, \mathcal{O}_X)$ be a scheme, which by abuse of notation is often shortened to X.

(1) In the case that X is a scheme over a field k, we say X is *geometrically* $\bullet$ if $X_{k'}$ is $\bullet$ for each field extension $k' \supset k$, where $\bullet$ can be any of the defined properties below.
(2) $(X, \mathcal{O}_X)$ is *irreducible* (resp. *quasicompact*, resp. *connected*) if X is irreducible (resp. quasicompact, resp. connected).
(3) X is *integral* if $X \neq \varnothing$ and for each affine open $\varnothing \neq \operatorname{Spec} R \subset X$, we have R is an integral domain.
(4) X is *reduced* if each $\mathcal{O}_{X,x}$ is reduced, or equivalently $\mathcal{O}_X(U)$ for each affine open U is reduced.
    X is integral if and only if X is reduced and irreducible.
(5) X is *normal* if each $\mathcal{O}_{X,x}$ is a normal domain, or equivalently $\mathcal{O}_X(U)$ for each affine open U is a normal ring.
(6) X is *locally noetherian* if for each $x \in X$, there is an affine open $\operatorname{Spec} R \ni x$ such that R is noetherian.
(7) X is *noetherian* if X is locally noetherian and quasicompact.
(8) X is *regular* or *nonsingular* if X is locally noetherian and each $\mathcal{O}_{X,x}$ is regular.

Definition 2.6 (Properties of a morphism of schemes)**.** Let $(\pi, f) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. By abuse of notation, (π, f) is often shortened to f.

(1) If the f is fixed or canonical, we say X is $\bullet$ over Y where $\bullet$ can be any of the defined adjectives below. Moreover, if $Y = \operatorname{Spec} A$ is affine and this base is fixed, we simply say X is $\bullet$.
    f is *universally* $\bullet$ if for any morphism $S \to Y$, the base change $f' : X \times_Y S \to S$ is $\bullet$, where $\bullet$ can be any of the defined properties below.
(2) (π, f) is a *closed immersion* if π is a homeomorphism of X with a closed subset of Y, and f is surjective. In this case we say X is a closed subscheme of Y.
(3) (π, f) is an *open immersion* if π is a homeomorphism of X with an open subset of Y, and f is an isomorphism. In this case we say X is an open subscheme of Y.
(4) f is an *immersion* if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion. In this case we say X is a *subscheme* of Y.
(5) (π, f) is *dominant* (resp. *surjective*, resp. *open*, resp. *closed*) if π is dominant (resp. surjective, resp. open, resp. closed).
(6) f is *affine* if for each affine open $U \subset Y$, we have $f^{-1}(U)$ is affine open in X.
(7) f is *quasicompact* if for each affine open $U \subset Y$, we have $f^{-1}(U)$ is quasicompact.
(8) f is *quasiseparated* (resp. separated) if $\delta_{X/Y}$ is quasicompact (resp. a closed immersion).
(9) f is *of finite type at* $x \in X$ if there are affine opens $\operatorname{Spec} A \ni x$ and $\operatorname{Spec} B \subset Y$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ and A is isomorphic to $B[x_1, \ldots, x_n]/I$ for some n and I.
(10) f is *locally of finite type* if f is of finite type at each $x \in X$.
(11) f is *of finite type* if f is locally of finite type and quasicompact.
(12) f is *of finite presentation at* $x \in X$ if it's of finite type at $x \in X$, with the extra hypothesis that I is finitely generated.
(13) f is *locally of finite presentation* if f is of finite presentation at each $x \in X$.
(14) f is *of finite presentation* if f is locally of finite presentation, quasicompact and quasiseparated.
(15) f is *flat at* $x \in X$ if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a flat ring map.
(16) f is *flat* if f is flat at each $x \in X$, is *faithfully flat* if f is moreover surjective, and is *fppf* (resp. *fpqc*) if f is moreover of finite presentation (resp. quasicompact).
(17) f is *smooth at* $x \in X$ (*of relative dimension* d) if there are affine opens $\operatorname{Spec} A \ni x$ and $\operatorname{Spec} B \subset Y$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, and an open immersion

$$\operatorname{Spec} A \to \operatorname{Spec} B[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-d})$$

for some n and f_i such that the Jacobian

$$\left(\frac{\partial f_i}{\partial x_j}(x) \right)_{ij}$$

has rank $n - d$. (In particular f is of finite presentation at x.)
(18) f is *smooth of relative dimension* d if f is smooth at each $x \in X$ of relative dimension d.
(19) f is *unramified at* $x \in X$ if f is of finite type at $x \in X$ and $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified.
(20) f is *unramified* if f is unramified at each $x \in X$.
(21) f is *étale* if f is smooth and unramified.
(22) f is *proper* if f is separated, of finite type, and universally closed.
(23) f is *integral* (resp. *finite*) if f is affine and for each affine open $\operatorname{Spec} A \subset Y$ with $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B \subset X$, the ring map $A \to B$ is integral (resp. finite).
    Every finite morphism of schemes is integral.

(24) In the case that each fibre X_y is a locally noetherian scheme, we say f is *normal* (resp. *regular*) if f is flat and X_y is geometrically normal (resp. regular).

Definition 2.7 (More properties for a scheme over a field k)**.** Let X be a scheme over k.

(1) *Algebraic* is another word for "of finite type over k" for X.
(2) X is a *variety* if X is separated and algebraic (sometimes geometrically reduced is also added).
(3) X is *complete* if X is proper over k.

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