# A GLOSSARY OF ALGEBRAIC GEOMETRY

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## 1. Algebra

**Definition 1.1** (Properties of a ring). Let $A$ be a ring.

(1) $A$ is *reduced* if for each $a \in A$, we have $a^2 = 0 \implies a = 0$ (that is, $A$ has no nonzero nilpotent elements).
(2) $A$ is *normal* if the localisation $R_\mathfrak{p}$ at each prime ideal $\mathfrak{p}$ of $R$ is a normal (integrally closed) domain, that is,

$$R_\mathfrak{p} = \{s \in \mathrm{Frac}(R_\mathfrak{p}) : \exists \text{ monic } g \in R_\mathfrak{p}[x] : g(s) = 0\}.$$

(3) In the case that $A$ is noetherian, we say $A$ is *regular* if each $R_\mathfrak{p}$ is regular, that is, the Krull dimension of $R_\mathfrak{p}$ equals the minimal number of generators of its maximal ideal.

**Definition 1.2** (Properties of a ring map). Let $f : A \to B$ be a ring map.

(1) $f$ is *flat* if the functor $- \otimes_A B : \mathsf{Mod}_A \to \mathsf{Mod}_A$ is exact, that is, $B_1 \to B_2 \to B_3$ is exact implies $B_1 \otimes_A B \to B_2 \otimes_A B \to B_3 \otimes_A B$ is exact for $A$-modules $B_1, B_2, B_3$.
(2) $f$ is *faithfully flat* if moreover the converse implication holds.
(3) $f$ is *integral* if for each $b \in B$, there is a monic $g \in A[x]$ such that $\overline{g}(b) = 0$ (there is a slight abuse of notation: the last $g$ denotes the image of the first $g$ in $B[x]$).
(4) $f$ is *of finite type* if there is a surjective $A$-algebra map $A[x_1, \dots, x_n] \to B$ for some $n \in \mathbb{N}$.
(5) $f$ is *of finite presentation* if there is an isomorphism of $A$-algebras $A[x_1, \dots, x_n]/(f_1, \dots, f_m) \cong B$ for some $n, m \in \mathbb{N}$ and $f_1, \dots, f_m \in A[x_1, \dots, x_n]$.
(6) $f$ is *finite* if $B$ is a finite $A$-module, that is, there a surjective $A^{\oplus n} \to B$ for some $n \in \mathbb{N}$. (Every finite ring map is integral.)
(7) Denote by $I$ the kernel of the natural surjection $A[B] \to B$. Call $I/I^2 \to \Omega_{A[B]/A} \otimes_{A[B]} B$ the *naive cotangent complex* and denote it by $\mathrm{NL}_{B/A}$. The kernel of it is then denoted by $H_1(\mathrm{NL}_{B/A})$.
(8) $f$ is *smooth* if $f$ is of finite presentation, $H_1(\mathrm{NL}_{B/A}) = 0$, and $\Omega_{B/A}$ is a finite projective $B$-module.
(9) In the case that $A$ and $B$ are local with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$, we say $f$ is a *local ring map* if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

**Definition 1.3.** A local ring map $f : A \to B$ with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ is *unramified* if $f(\mathfrak{m}_A)B = \mathfrak{m}_B$, and $B/\mathfrak{m}_B$ is a finite separable extension of $A/\mathfrak{m}_A$.

## 2. Geometry

**Definition 2.1.** Let $(X, O_X)$ be a ringed space. $(X, O_X)$ is a *locally ringed space* if each stalk $O_{X,x}$ is a local ring. In this case we denote the (unique) maximal ideal of $O_{X,x}$ by $\mathfrak{m}_x$, and the residue field by $\kappa(x)$.

A *morphism of locally ringed spaces* $(\pi, f) : (X, O_X) \to (Y, O_Y)$ is a pair where $\pi : X \to Y$ is a continuous map of topological spaces and $f : O_Y \to \pi_* O_X$ is a natural transformation of functors, such that the induced map $f_x : O_{Y,f(x)} \to O_{X,x}$ of stalks is a local ring map.

**Definition 2.2.** A *scheme* is a ringed space locally isomorphic to affine $(\operatorname{Spec} A, O_{\operatorname{Spec} A})$. They are locally ringed spaces. A *morphism of schemes* is a morphism of them as locally ringed spaces.

An $S$-scheme refers to a scheme $X$ with a fixed morphism $X \to S$, and for a ring $A$, an $A$-scheme is a $\operatorname{Spec} A$-scheme.

**Definition 2.3.** Let $f : X \to Y$ be a morphism of schemes. The *diagonal morphism* of $f$ is the unique morphism $\partial_{X/Y} : X \to X \times_Y X$ such that $\mathrm{pr}_1 \circ \partial_{X/Y} = \mathrm{pr}_2 \circ \partial_{X/Y} = \mathrm{id}_X$.

**Definition 2.4.** Let $(X, O_X)$ be a scheme and $U \subset X$ an open subset. Then $(U, O_{X|U})$ is a scheme, and we say $U$ is *affine open* in $X$ if this $(U, O_{X|U})$ is affine.

**Definition 2.5** (Properties of a scheme)**.** Let $(X, O_X)$ be a scheme, which by abuse of notation is often shortened to $X$.

(1) In the case that $X$ is a scheme over a field $k$, we say $X$ is *geometrically* • if $X_{k'}$ is • for each field extension $k' \supset k$, where • can be any of the defined properties below.

(2) $(X, O_X)$ is *irreducible* (resp. *quasicompact*, resp. *connected*, resp. *catenary*) if $X$ is irreducible (resp. quasicompact, resp. connected, resp. catenary (i.e. every pair of irreducible closed subsets $T \subset T'$ admits a maximal chain of irreducible closed subsets $T = T_0 \subset T_1 \subset \cdots \subset T_e = T'$ and any such chain has the same length)).

(3) $X$ is *integral* if $X \neq \emptyset$ and for each affine open $\emptyset \neq \operatorname{Spec} R \subset X$, we have $R$ is an integral domain.

(4) $X$ is *reduced* if each $O_{X,x}$ is reduced, or equivalently $O_X(U)$ for each affine open $U$ is reduced.
$X$ is integral if and only if $X$ is reduced and irreducible.

(5) $X$ is *normal* if each $O_{X,x}$ is a normal domain, or equivalently $O_X(U)$ for each affine open $U$ is a normal ring.

(6) $X$ is *locally noetherian* if for each $x \in X$, there is an affine open $\operatorname{Spec} R \ni x$ such that $R$ is noetherian.

(7) $X$ is *noetherian* if $X$ is locally noetherian and quasicompact.

(8) $X$ is *regular* or *nonsingular* if $X$ is locally noetherian and each $O_{X,x}$ is regular.

(9) $X$ is *factorial* if each $O_{X,x}$ is a unique factorisation domain.

**Definition 2.6** (Properties of a morphism of schemes)**.** Let $(\pi, f) : (X, O_X) \to (Y, O_Y)$ be a morphism of schemes. By abuse of notation, $(\pi, f)$ is often shortened to $f$.

(1) If the $f$ is fixed or canonical, we say $X$ is • over $Y$ where • can be any of the defined adjectives below. Moreover, if $Y = \operatorname{Spec} A$ is affine and this base is fixed, we simply say $X$ is •.
$f$ is *universally* • if for any morphism $S \to Y$, the base change $f' : X \times_Y S \to S$ is •, where • can be any of the defined properties below.

(2) $(\pi, f)$ is a *closed immersion* if $\pi$ is a homeomorphism of $X$ with a closed subset of $Y$, and $f$ is surjective. In this case we say $X$ is a closed subscheme of $Y$.

(3) $(\pi, f)$ is an *open immersion* if $\pi$ is a homeomorphism of $X$ with an open subset of $Y$, and $f$ is an isomorphism. In this case we say $X$ is an open subscheme of $Y$.

(4) $f$ is an *immersion* if it can be factored as $j \circ i$ where $i$ is a closed immersion and $j$ is an open immersion. In this case we say $X$ is a *subscheme* of $Y$.

(5) $(\pi, f)$ is *dominant* (resp. *surjective*, resp. *open*, resp. *closed*) if $\pi$ is dominant (resp. surjective, resp. open, resp. closed).
(6) $f$ is *affine* if for each affine open $U \subset Y$, we have $f^{-1}(U)$ is affine open in $X$.
(7) $f$ is *quasicompact* if for each affine open $U \subset Y$, we have $f^{-1}(U)$ is quasicompact.
(8) $f$ is *quasiseparated* (resp. separated) if $\delta_{X/Y}$ is quasicompact (resp. a closed immersion).
(9) $f$ is *of finite type at* $x \in X$ if there are affine opens $\operatorname{Spec} A \ni x$ and $\operatorname{Spec} B \subset Y$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ and $B \to A$ is a ring map of finite type.
(10) $f$ is *locally of finite type* if $f$ is of finite type at each $x \in X$.
(11) $f$ is *of finite type* if $f$ is locally of finite type and quasicompact.
(12) $f$ is *of finite presentation at* $x \in X$ if it's of finite type at $x \in X$ and the corresponding $B \to A$ is moreover of finite presentation.
(13) $f$ is *locally of finite presentation* if $f$ is of finite presentation at each $x \in X$.
(14) $f$ is *of finite presentation* if $f$ is locally of finite presentation, quasicompact and quasiseparated.
(15) $f$ is *flat at* $x \in X$ if $O_{Y,f(x)} \to O_{X,x}$ is a flat ring map.
(16) $f$ is *flat* if $f$ is flat at each $x \in X$, is *faithfully flat* if $f$ is moreover surjective, and is *fppf* (resp. *fpqc*) if $f$ is moreover of finite presentation (resp. quasicompact).
(17) $f$ is *smooth at* $x \in X$ (*of relative dimension* $d$) if there are affine opens $\operatorname{Spec} A \ni x$ and $\operatorname{Spec} B \subset Y$ such that $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$, and an open immersion

$$\operatorname{Spec} A \to \operatorname{Spec} B[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-d})$$

for some $n$ and $f_i$ such that the Jacobian

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{ij}$$

has rank $n - d$. (In particular $f$ is of finite presentation at $x$.) Or, if $B \to A$ is a smooth ring map and $\Omega_{X/Y}$ is locally free of constant rank $d$.
(18) $f$ is *smooth of relative dimension* $d$ if $f$ is smooth at each $x \in X$ of relative dimension $d$.
(19) $f$ is *unramified at* $x \in X$ if $f$ is of finite type at $x \in X$ and $O_{Y,f(x)} \to O_{X,x}$ is unramified.
(20) $f$ is *unramified* if $f$ is unramified at each $x \in X$.
(21) $f$ is *étale* if $f$ is smooth and unramified.
(22) $f$ is *proper* if $f$ is separated, of finite type, and universally closed.
(23) $f$ is *integral* (resp. *finite*) if $f$ is affine and for each affine open $\operatorname{Spec} A \subset Y$ with $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B \subset X$, the ring map $A \to B$ is integral (resp. finite).
Every finite morphism of schemes is integral.
(24) In the case that each fibre $X_y$ is a locally noetherian scheme, we say $f$ is *normal* (resp. *regular*) if $f$ is flat and $X_y$ is geometrically normal (resp. regular).

**Definition 2.7** (More properties for a scheme over a field $k$)**.** Let $X$ be a scheme over $k$.

(1) *Algebraic* is another word for "of finite type over $k$" for $X$.
(2) $X$ is a *variety* if $X$ is separated and algebraic (sometimes geometrically reduced is also added).
(3) $X$ is *complete* if $X$ is proper over $k$.

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