FAMILIES OF ALGEBRAIC CYCLES – CHOW SCHEMES

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ABSTRACT. The following is a English translation and LaTeXitification of B. ANGÉNIOL. *Familles de cycles algébriques – schéma de Chow*. Lecture Notes in Mathematics 896. Berlin, Heidelberg, New York: Springer, 1981. Zbl: 0496.14004. We claim no originality except errors.

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0. Introduction

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The aim of this work is to endow the set of compact cycles of a given scheme with the structure of an algebraic space. This is achieved here in characteristic zero.

More precisely, let K be a field and let X be a smooth scheme over K of pure dimension $N = n + p$. Let $\xi^p(X)$ be the free group $\mathbb{Z}^{(X^p)}$ whose basis is the set of proper irreducible closed subsets of codimension p in X, ordered by the order induced by the product order on $\mathbb{Z}^{X^p}$. We denote by $C^p(X)$ the set of positive elements of $\xi^p(X)$. The elements of $C^p(X)$ are called cycles purely of codimension p on X. The goal is therefore to endow $C^p(X)$ with the structure of an algebraic space. To define a family of cycles of X parametrised by a K-scheme S, it's most natural to consider a cycle Z of $X \times S$, and to impose on the cycles Z_s, as s ranges over S, regularity conditions.

A natural regularity condition is flatness: to a cycle Z, one associates a scheme structure taking multiplicities into account, and one then requires that the morphism from Z to S be flat. This is what led to the construction of the Hilbert scheme (Douady space). This condition is very reasonable if one assumes the parameter space S to be smooth, since it suffices, for example, that Z be Cohen–Macaulay for the morphism from Z to S to be flat. However, if S is not smooth, even for 0-cycles, one loses entirely natural families of cycles, as the following example shows.

Take $X = \mathbb{A}^2_k = \operatorname{Spec}(K[t,u])$ and $S = \operatorname{Spec}(K[x,y,z]/(xy - z^2))$. One then has a parametrisation of S by the points of X via $x = t^2,\ y = u^2,\ z = tu$. This defines a finite morphism from X to S. Above a point s of the cone S distinct from the vertex, there are two points of X with coordinates (t,u) and $(-t,-u)$, and above the vertex s_0 of S lies the origin $(0,0)$ of X. One is therefore tempted to say that the family of 0-cycles of X parametrised by S, consisting above $s \neq s_0$ of the sum of the points with coordinates (t,u) and $(-t,-u)$, and above s_0 of 2 times the point with coordinates $(0,0)$, is an algebraic family. p. 2

Yet the morphism f from X to S is not flat. To see this, since f is finite and S is integral, it suffices to verify that the integer $\dim_{k(s)}\big(f_*(O_X)_s \otimes k(s)\big)$ is not independent of the point s. If s is distinct from s_0, the fibre above s is a union of two simple points and the integer above is therefore equal to 2, whereas at the point s_0 one obtains $\dim\big(K[t,u]/(t^2, tu, u^2)\big) = 3$, whence the result.

However, in this example, the cycles under consideration are cycles of dimension 0 and degree 2. Such a cycle may therefore be written $P_1 + P_2$ where the P_i are points of X (we assume for simplicity that K is algebraically closed). The space of such cycles is therefore the set of unordered pairs (P_1, P_2), namely $X \times X/\mathfrak{S}_2$, where $\mathfrak{S}_2$ is the symmetric group of order 2, which we shall denote $\operatorname{Sym}^2 X$. To prove that the family under consideration is indeed algebraic, one must therefore show that it corresponds to a morphism from S to $\operatorname{Sym}^2 X$.

To this end, let us study the equations of $\operatorname{Sym}^2 X$. We shall embed $\operatorname{Sym}^2 X$ in $\mathbb{A}^5_K$ via Newton coordinates: that is, considering two copies of X with coordinates (t,u) and (t',u'), one sets $N_1 = t + t'$, $N_2 = u + u'$, $N_{11} = t^2 + t'^2$, $N_{12} = tu + t'u'$, $N_{22} = u^2 + u'^2$. One then has

Lemma. $\operatorname{Sym}^2 X$ is isomorphic to Spec $(K[N_1, N_2, N_{11}, N_{12}, N_{22}]/(h))$, where h is the determinant

$$\begin{vmatrix} 2 & N_1 & N_2 \\ N_1 & N_{11} & N_{12} \\ N_2 & N_{12} & N_{22} \end{vmatrix}.$$

The morphism from $\operatorname{Sym}^2 X$ to $\mathbb{A}^5_K$ is indeed an embedding, since we are in characteristic p. 3
0 and the Newton symmetric functions of order at most 2 generate all symmetric functions. Moreover, if the equation above is satisfied, there exist t and t' such that $N_1 = t+t'$, $N_{11} = t^2+t'^2$, and there exist u and u' such that $N_2 = u + u'$, $N_{22} = u^2 + u'^2$, and one has

$$N_{12}^2 - [(tu + t'u') + (tu' + t'u)]N_{12} + (tu + t'u')(tu' + t'u) = 0,$$

whence $N_{12} = tu + t'u'$ or $tu' + t'u$, which proves the lemma (upon swapping u and u').

Since the morphism from X to S is finite, we shall associate to it a trace map θ from O_X to O_S given by the formula

$$\theta(g)(s) = g(P_1) + g(P_2),$$

where g is a function on X, s a point of S, and P_1 and P_2 the two points of X lying above s. One then has:

$$\theta(t) = t + (-t) = 0,\ \theta(u) = u + (-u) = 0,\ \theta(t^2) = t^2 + (-t)^2 = 2t^2 = 2x,$$
$$\theta(u^2) = u^2 + (-u)^2 = 2y,\ \theta(tu) = tu + (-t)(-u) = 2tu = 2z.$$

The mapping from S into $\operatorname{Sym}^2 X$ will therefore correspond to the ring morphism given by $N_1 \to \theta(t),\ N_2 \to \theta(u),\ N_{11} \to \theta(t^2),\ N_{12} \to \theta(tu),\ N_{22} \to \theta(u^2)$, so that it suffices to verify that the determinant $\begin{vmatrix} 2 & \theta(t) & \theta(u) \\ \theta(t) & \theta(t^2) & \theta(tu) \\ \theta(u) & \theta(tu) & \theta(u^2) \end{vmatrix}$ is equal to 0. Now, this determinant is equal to $8(xy - z^2) = 0$ on S, whence the result.

Let us return to the general problem.

When $n = 0$, a cycle can be written $\sum_{i=1}^{\ell} n_i P_i$, where $n_i \in \mathbb{N}$, and where the P_i are closed points of X. If $\sum_{i=1}^{\ell} n_i \dim_K k(P_i) = k$, such a cycle can therefore be identified with the point of $\operatorname{Sym}^k(X)$ composed of the point P_1 (n_1 times), etc..., of the point P_ℓ (n_ℓ times) ($\operatorname{Sym}^k(X)$ is the quotient of the product X^k by the action of the symmetric group $\mathfrak{S}_k$). We thus immediately p. 4
have a structure of an algebraic space of finite type on the space of cycles of dimension 0 of degree k, hence a structure of an algebraic space locally of finite type on $C^N(X)$.

In the case where X is equal to the projective space (or by extension a projective variety), cycles are identified by "Chow coordinates". The idea is to consider the intersection of a cycle of $C^p(X)$ with a plane of dimension $p - 1$. This intersection is generally empty, and the set of planes of dimension $p - 1$ which meet a given cycle constitutes a hypersurface of the Grassmannian; this allows the cycle to be identified by means of the coefficients of the equation of the hypersurface (the Cayley form), which are what are called the Chow coordinates. Unfortunately, this construction is based on the vacuity of intersection of varieties and thus does not lend itself to a satisfactory study of infinitesimal variations of cycles. Moreover, it rests on global arguments which link it to the projective case and render it nonintrinsic.

If S is a scheme over K, to define a family of cycles of X parametrised by S, the fundamental idea of Barlet in [2] was to posit that a family $(Z_s)_{s\in S}$ of $C^p(X)$ was algebraic if every time one intersected $(Z_s)_{s\in S}$ with a plane of dimension p in such a way that the intersection was finite for all s, one obtained a locally algebraic family of dimension zero (thus corresponding to a morphism $S \to \operatorname{Sym}^k X$). This local formulation allows one to dispense with projective hypotheses by positing that a family $(Z_s)_{s\in S}$ is algebraic if for every local projection onto a smooth S-scheme B of relative dimension n, such that Z is quasifinite over B, the cycle Z corresponds to a morphism

$B \to \mathrm{Sym}^k_B(X \times S)$ (quotient of $(X \times S) \times_B (X \times S) \cdots \times_B (X \times S)$ by $\mathfrak{S}_k$). But the difficulty is then the change of projection: when can one say that two morphisms $B \to \mathrm{Sym}^k_B(X \times S)$ and $B' \to \mathrm{Sym}^k_{B'}(X \times S)$ correspond to the same cycle? p. 5

Here again, one must restrict oneself to the case where S is reduced, which allows the previous question to be answered by taking into account only the supports and the multiplicities. Furthermore, let us note that a point of $\mathrm{Sym}^k(X)$ is identified by the values taken at this point by the symmetric functions; one can only consider it as an unordered k-tuple of points of X with values in K if K is algebraically closed.

The approach that we follow here is to take up the ideas of Barlet by addressing, according to a suggestion of Deligne, the problem of the change of projection from the viewpoint of the theory of duality and of fundamental classes developed in [5] and [3]. In clear terms, observing that the morphisms $B \to \mathrm{Sym}^k_B(X\times S)$ correspond bijectively to certain traces $\theta : O_{X\times S} \to O_B$, and that a class c in $H^p_Z\left(X \times S, \Omega^p_{X\times S/S}\right)$ induces for every projection a trace $O_{X\times S} \to O_B$, we shall say that two morphisms $B \to \mathrm{Sym}^k_B(X \times S)$ and $B' \to \mathrm{Sym}^k_{B'}(X \times S)$ correspond to the same cycle if they originate from the same class. Let us note here that we have identified here the elements of $\mathrm{Sym}^k(X)$ by means of traces, that is to say, Newton's symmetric functions; as these generate all symmetric functions only in characteristic zero, this leads us therefore to assume K to be of characteristic zero. The existence of a crystalline duality theory (with divided powers) is lacking for an attempt to treat the case of positive characteristic. Thus, a family of compact cycles of X parametrised by S will be a pair (Z, c), where Z is a proper closed subset of $X \times S$ of pure codimension p fibre by fibre, and c a class of $H^p_Z\left(X \times S, \Omega^p_{X\times S/S}\right)$, nonzero at the generic points of the irreducible components of Z, and "admissible", that is to say, satisfying certain properties. Recall (cf. (2.1.2.1), (2.1.3.1)), that a class c corresponds to the datum, for an arbitrary projection onto a scheme B smooth over S such that Z is finite over B, of a trace $\theta : O_{Z_m} \otimes \Omega^\bullet_{X\times S/S} \to \Omega^\bullet_{B(S)}$ where Z_m is the mth infinitesimal neighbourhood of Z, and m is sufficiently large. p. 6

Let us call an admissible family of classes, denoted by $\mathfrak{J}$, the datum, for every scheme S and every closed proper subset $Z \subset X \times S$ of pure codimension p in each fibre of the morphism $X \times S \to S$, of a set of classes c in $H^p_Z\left(X \times S, \Omega^p_{X\times S/S}\right)$, such that the following properties are satisfied:

a) *Preservation of multiplicity*: A class c of $\mathfrak{J}$ is closed for d' (which is equivalent to saying that the trace θ commutes with the differentials).
b) *Normalisation*: If Z is an S-smooth, irreducible scheme, and if c is a class of $\mathfrak{J}$ inducing multiplicity 1 on Z, then c is the fundamental class of Z ([3], [4]).
c) *Specialisation*: For every affine scheme U étale over $X \times S$, and every projection onto a smooth scheme B, let $Y_{U,B}$ be the space of classes of $\varinjlim_Z H^p_Z\left(U, \Omega^p_{U/S}\right)$ with support Z of pure codimension p in each fibre over S, and such that the reduced scheme underlying this closed set is finite over B; if $k \in \mathbb{N}$, let E_k be the subset of $Y_{U,B}$ formed of classes which can be written as a sum of k fundamental classes of smooth S-schemes of degree 1 over B (which is equivalent to saying that the trace θ can be written as a sum of k algebra morphisms). The trace corresponding via (2.1.2.1) and (2.1.3.1) to a class c of $\mathfrak{J}$ satisfies the natural equations (cf. 1.5.0.3 for the meaning) satisfied by the traces associated with the classes of E_k, that is to say, with the fundamental classes of the finite étale coverings of B.
d) *Localisation*: A class c belongs to $\mathfrak{J}$ if and only if for every point z of Z, there exists a neighbourhood U of z in $X \times S$ for the étale topology, such that c_U belongs to the inverse image $\mathfrak{J}_U$ of $\mathfrak{J}$ on U.

Consequently, if one denotes by $\Omega^{\bullet\sigma}_{U^k/S}$ the subalgebra of $\Omega^{\bullet}_{U\times_B U\times_B\cdots\times_B U/S}$ formed of the p. 7
differentials invariant under the action of $\mathfrak{S}_k$, one can set:

Definition. A class c of $H^p_Z\left(X\times S,\Omega^p_{X\times S/S}\right)$ is a *Chow class* on U for a projection $U\longrightarrow B$ finite on Z if it induces a morphism $\Omega^{\bullet\sigma}_{U^k/S}\to\Omega^{\bullet}_{B/S}$ compatible with the exterior differential and the operations of contraction on the successive tangent spaces (cf. (4.1.0.2) for a precise definition).

The class c is a Chow class if it is a Chow class on all the open sets of a covering of $X\times S$ (for the étale topology).

We then have the following:

Theorem. The family of Chow classes is an admissible family of classes.

If we denote by $C^p_X(S)$ the set of families of compact cycles parametrised by S, $S\mapsto C^p_X(S)$ is a contravariant functor representable in the category of algebraic spaces; this therefore indeed endows $C^p(X)$ with a structure of an algebraic space.

In Chapter 1, we define the Wäring multilinear forms, and we characterise the traces which induce a ring morphism on the symmetric tensors: $\Omega^{\bullet\sigma}_{U^k/S}\to\Omega^{\bullet}_{B/S}$. This leads us to provide explicitly the equations of $\mathrm{Sym}^k(\mathbb{C}^p)$ for the embedding by Newton's symmetric functions.

In Chapters 2 and 3, we provide the residue isomorphism explicitly and express the trace associated with a class c in a projection in terms of the trace associated with this same class in another projection; then, we express in terms of the trace in one projection the fact that for every projection, the trace satisfies a family of equations. We apply this to the case of Wäring multilinear forms, and we obtain a family of equations independent of the projection. Then p. 8
we show that a finite number of these equations generates the others.

In Chapter 4, we define what a Chow class is; we deduce the definition of the Chow functor and study its tangent space.

In Chapter 5, we show that the Chow functor is representable in the category of algebraic spaces.

In Chapter 6, we show that the reduced space associated with the Chow algebraic space coincides with Barlet's space of cycles, and thus in the projective case with the Chow coordinates. We provide a theorem of direct image of cycles which allows $C^p(X)$ to be defined even when X is not smooth. We conclude by providing explicitly the local equations of the scheme of curves of degree 2 in $\mathbb{P}^3(\mathbb{C})$.

In Chapter 7, we define a morphism from the Hilbert scheme into the Chow scheme, and we show that it is an isomorphism in the case of divisors ($p=1$). To this end, we must first provide an explicit formula for the trace $\Omega^{\bullet}_{Z/S}\to\Omega^{\bullet}_{B/S}$ in the case where Z is a finite and flat scheme over a smooth scheme B (the existence of this trace had been demonstrated in [1]).

Finally, in Chapter 8, we define the intersection of two algebraic families of cycles, and we characterise algebraic equivalence in terms of the connected components of $C^p(X)$.

1. Wäring multilinear forms p. 9

Let $(x_i)_{i\in I}$ be a family of k points of a vector space V over $\mathbb{C}$. Let S_h (resp. N_h) be the element of $S^h(V)$ (the hth component of the symmetric algebra of V) equal to the hth elementary symmetric function of the x_i (resp. the hth Newton symmetric function in the x_i). We show that there exist universal expressions P^k with coefficients in $\mathbb{Z}$ called Wäring multilinear forms, such that we have $h!S_h=P^h(N_1;\ldots;N_h)$, and such that conversely, given elements N'_h of $S^h(V)$ for all $h\in\mathbb{N}$, these are the Newton symmetric functions of elements $x'_1,\ldots,x'_k$ of V if and only if we have: $N'_0=k$ and for all $i_1,i_2,\ldots,i_{k+1}\in\mathbb{N}$, $P^{k+1}\left(N'_{i_1};\ldots;N'_{i_{k+1}};\ldots;N'_{i_1+\cdots+i_{k+1}}\right)=0$, where this expression is the polarised expression associated with P^{k+1} (cf. Theorem 1.5.0.3). More generally, we show that, given a linear form θ on V^*, in order for there to exist a family of k points $(x_i)_{i\in I}$ such that $(\forall f \in V^*,\ \theta(f) = \sum_{i\in I} f(x_i))$, it is necessary and sufficient that we have $P_\theta^{k+1} = 0$ (1.5.0.3). We even treat here the case of a morphism of A-modules $\theta : B \longrightarrow A$, where A is any ring of characteristic zero, and B is an A-algebra.

1.1. Rules of signs.

1.1.1. In this entire section, we shall place ourselves in a $\mathbb{Q}$-linear abelian category $\mathcal{A}$, endowed with a tensor product $\otimes$, which is associative, commutative, and unitary, such that the functors $X \longrightarrow A \otimes X$ are additive and right exact for every A in $\mathcal{A}$.

1.1.2. *Examples.*

1.1.2.1.

1.1.3. *Language.*

1.2. Tenseurs symétriques.

1.3. Complexe semi-simplicial associé à θ.

1.4. Calcul des formes multilinéaires de Wäring.

1.5. La trace universelle.

1.5.0.1. **Definition.**

1.5.0.2. **Proposition.**

1.5.0.3. **Theorem.**

1.6. Revêtement canonique de $\mathrm{TS}_A^k(B)$.

1.7. Sommes de traces et traces de traces.

1.8. Sur les équations de $\mathrm{TS}_A^k(B)$.

2. Classes de cohomologie locale et résidus

2.1. Résidus et traces de différentielles.

2.1.1.

2.1.2.

2.1.2.1. **Proposition.**

2.1.3.

2.1.3.1. **Proposition.**

2.2. Changement de projection infinitésimal.

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3. Invariance des formes multilinéaires de Wäring

3.1. Énoncé.

3.2. Interprétation géométrique : les espaces tangents symétriques.

3.3. Finitude.

3.4. Finitude et espaces tangents symétriques (bis).

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4.1. Classes de Chow.

4.1.0.1. Theorem.

4.1.0.2. Definition.

4.2. Foncteur de Chow.

4.3. Espace tangent – déformations.

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5.2. Schéma de Chow.

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7.1. Classe fondamentale dans le cas plat.

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8. Intersections

8.1. Cup-produits et intersections.

8.2. Équivalence algébrique. Images réciproques.

8.3. Intégration de classes de cohomologie.

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