COMPLEX MANIFOLDS WITHOUT POTENTIAL THEORY (WITH AN APPENDIX ON THE GEOMETRY OF CHARACTERISTIC CLASSES)

SHIING-SHEN CHERN

Typesetter's note. The following is a LaTeXitification of S. Chern. *Complex manifolds without potential theory (with an appendix on the geometry of characteristic classes)*. 2nd ed. Universitext. New York, NY: Springer, 1979. Zbl: 0444.32004. We mostly respect the original typography, but we use British spellings and do not capitalise name-derived adjectives, e.g. kählerian. For convenience, we number the examples, propositions etc. by sections via amsthm. Similarly, originally the bibliography is separated between the main text and the appendix, and further categorised into books, articles and others added during second edition; for simplicity, we put all references in one section and let BibLaTeX order them, and we add bibliographic information to help the reader navigate to the source. We claim no originality except errors.

Preface. The main text, on complex manifolds, was the notes from a course with the same title given at UCLA in the fall of 1966. It was written up after each lecture; only minor changes have been made. To the Department of Mathematics at UCLA, and Lowell Paige in particular, I wish to express here my belated thanks. The Appendix was an expanded version of a series of lectures given at a summer seminar of the Canadian Mathematical Congress taken place in Halifax, Nova Scotia in 1971. I wish to thank Professor J. R. Vanstone for his hospitality and kindness. Noteworthy is the treatment of the secondary characteristic classes, which is different from the one given in [2] in the Bibliography to the Appendix.

Needless to say, my deepest gratitude is to the University of California at Berkeley and the National Science Foundation for their continuous support of my research.

Contents

1. Introduction and examples 2
2. Complex and hermitian structures on a vector space 5
3. Almost complex manifolds; integrability conditions 9
4. Sheaves and cohomology 16
5. Complex vector bundles; connections 21
6. Holomorphic vector bundles and line bundles 31
7. Hermitian geometry and kählerian geometry 37
8. The Grassmann manifold 45
9. Curves in a Grassmann manifold 54

Appendix: Geometry of characteristic classes 55
1. Historical remarks and examples 55
2. Weil homomorphism 55
3. Secondary invariants 55
4. Vector fields and characteristic numbers 55
5. Holomorphic curves 55
References 55
Index 55

1. Introduction and examples

A complex manifold is a paracompact Hausdorff space which has a covering by neighbourhoods each homeomorphic to an open set in the m-dimensional complex number space such that where two neighbourhoods overlap the local coordinates transform by a complex analytic transformation. That is, if $z^1, \dots, z^m$ are local coordinates in one such neighbourhood and if $w^1, \dots, w^m$ are local coordinates in another neighbourhood, then where they are both defined, we have $w^i = w^i\left(z^1, \dots, z^m\right)$, where each w^i is a holomorphic (or analytic) function of the z's and the functional determinant $\frac{\partial\left(w^1,\dots,w^m\right)}{\partial\left(z^1,\dots,z^m\right)} \neq 0$. p. 1

We will give some examples of complex manifolds:

1.1. **Example.** The complex number space $\mathbb{C}_m$ whose points are the ordered m-tuples of complex numbers $\left(z^1, \dots, z^m\right)$. $\mathbb{C}_1$ is called the gaussian plane.

1.2. **Example.** The complex projective space $\mathbb{P}_m$. To define it, take $\mathbb{C}_{m+1} - 0$, where 0 is the point $(0, \dots, 0)$, and identify those points $\left(z^0, z^1, \dots, z^m\right)$ which differ from each other by a factor. The resulting quotient space is $\mathbb{P}_m$. It can be covered by $m+1$ open sets U_i defined respectively by $z^i \neq 0$, $0 \leq i \leq m$. In U_i we have the local coordinates ${}_i\zeta^k = \frac{z^k}{z^i}$, $0 \leq k \leq m$, $k \neq i$. The transition of local coordinates in $U_i \cap U_j$ is given by ${}_j\zeta^h = \frac{{}_i\zeta^h}{{}_i\zeta^j}$, $0 \leq h \leq m$, $h \neq j$, which are holomorphic functions. In particular, $\mathbb{P}_1$ is the Riemann sphere.

By assigning to a point of $\mathbb{C}_{m+1} - 0$ the point it defines in the quotient space, we get a natural projection $\psi : \mathbb{C}_{m+1} - 0 \longrightarrow \mathbb{P}_m$, for which the inverse image of each point is $\mathbb{C}^* = \mathbb{C}_1 - 0$. This relationship is the first example of the important notion of a holomorphic line bundle and it is justified to enter into some detail. In fact, in $\psi^{-1}(U_i)$ we can use instead of the coordinates $\left(z^0, \dots, z^m\right)$ the coordinates ${}_i\zeta^h = \frac{z^h}{z^i}$, $0 \leq h \leq m$, $h \neq i$, and z^i. This has the advantage of expressing clearly the fact that $\psi^{-1}(U_i)$ is a product $U_i \times \mathbb{C}^*$, z^i being the fibre coordinate (relative to U_i). In $\psi^{-1}(U_i \cap U_j)$, the fibre coordinates z^i and z^j, relative to U_i and U_j respectively, are related by p. 2

$$z^i = z^j\,{}_j\zeta^i = \frac{z^j}{{}_i\zeta^j}.$$

Thus the change of fibre coordinates is expressed by the multiplication of a nonzero holomorphic function. The general notion of a holomorphic line bundle, which generalises this example, plays a central role in complex manifolds.

To a point $p \in \mathbb{P}_m$ the coordinates of a point of $\psi^{-1}(p)$ are called its homogeneous coordinates. They can be normalised so that

$$\sum z^k \bar{z}^k = 1. \tag{1.1}$$

Equation (1.1) defines a sphere S^{2m+1} of real dimension $2m + 1$.The restriction of ψ gives the mapping $\psi : S^{2m+1} \to \mathbb{P}_m$, under which the inverse image of each point is a circle. This is called the *Hopf fibring* of S^{2m+1}.

Further examples are obtained from submanifolds of $\mathbb{P}_m$ and quotient manifolds of $\mathbb{C}_m$.

1.3. **Example.** Nonsingular submanifolds of $\mathbb{P}_m$, in particular, the nonsingular hyperquadric

$$(z^0)^2 + \cdots + (z^m)^2 = 0. \tag{1.2}$$

By a theorem of Chow, every compact submanifold embedded in $\mathbb{P}_m$ is an algebraic variety, i.e. it is the locus defined by a finite number of homogeneous polynomial equations [5, p. 170].

It will not be significant to consider compact submanifolds of $\mathbb{C}_m$, because of the: p. 3

Theorem 1(A). A connected compact submanifold of $\mathbb{C}_m$ is a point.

The proof makes use of the lemma: Let f be a holomorphic function on a complex manifold M. Suppose $p_0 \in M$ is a point such that $|f(p)| \leq |f(p_0)|$ for all p in a neighbourhood of p_0. Then $f(p) = f(p_0)$ in a neighbourhood of p_0.

For one variable this follows from the maximum modulus principle. The case of m variables follows from the consideration of the lines through p_0 and the application of the one variable case to these lines.

Now let M be a connected compact submanifold of $\mathbb{C}_m$. Each coordinate of $\mathbb{C}_m$ is a holomorphic function on M. By the lemma, it must be constant on every connected component of M. Since M is connected, M is a point.

However, various significant examples arise from the quotient manifolds of $\mathbb{C}_m$.

1.4. **Example.** Let Γ be the discontinuous group generated by $2m$ translations of $\mathbb{C}_m$, which are linearly independent over the reals. Then $\mathbb{C}_m/\Gamma$ is called the *complex torus*. If a complex torus can be embedded as a nonsingular submanifold of a projective space of sufficiently high dimension, it is called an *abelian variety*.

Let Δ be the discontinuous group generated by $z^k \to 2z^k$, $1 \leq k \leq m$. The quotient manifold $(\mathbb{C}_m - 0)/\Delta$ is called the *Hopf manifold*. It is homeomorphic to $S^1 \times S^{2m-1}$.

Consider $\mathbb{C}_3$ to be the group of all matrices

$$\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

Let D be the discrete subgroup consisting of those matrices for which z_1, z_2, z_3 are gaussian integers (i.e. $z_k = m_k + in_k$, $1 \le k \le 3$ where m_k, n_k are rational integers). Then $\mathbb{C}_3/D$ is called an *Iwasawa manifold*. Its fundamental group is isomorphic to D, and hence is not abelian. p. 4

1.5. **Example.** An orientable surface is a complex manifold (of dimension one). We suppose the surface to be C^∞ and define on it a positive definite riemannian metric. By the theorem of Korn–Lichtenstein there exist local isothermal parameters x, y so that locally the metric can be written

$$\mathrm{d}s^2 = \lambda^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \quad \lambda > 0 \tag{1.4}$$

or $\mathrm{d}s^2 = \lambda^2 \mathrm{d}z\mathrm{d}\bar{z}$, where $z = x + iy$, the orientation of the manifold being defined by $\mathrm{d}dx \wedge \mathrm{d}y = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$. If w is another local coordinate we will have

$$\mathrm{d}s^2 = \lambda^2 \mathrm{d}z\, \mathrm{d}\bar{z} = \mu^2 \mathrm{d}w\, \mathrm{d}\bar{w}$$

because $\mathrm{d}s^2$ is globally defined. It follows that $\mathrm{d}w$ is a multiple of $\mathrm{d}z$ or $\mathrm{d}\bar{z}$. If we assume that the complex coordinates z and w define the same orientation, then $\mathrm{d}w$ must be a multiple of $\mathrm{d}z$. This means that w is a holomorphic function of z, and the surface becomes a complex manifold.

A one-dimensional complex manifold is called a *Riemann surface*.

1.6. **Example** (Calabi–Eckmann)**.** Let S and S' be spheres of dimensions $2p+1$ and $2q+1$ respectively, $p, q > 0$. By the Hopf fibring in Ex. 1.2 we have a fibration

$$\pi : S \times S' \longrightarrow \mathbb{P}_p \times \mathbb{P}_q$$

with fibre a two (real) dimensional torus. Since both the base space and the fibre are complex manifolds, we would expect that the total space could be given a complex structure. This we will prove to be the case as follows: p. 5

Let S be the set of all points $z = \left(z^0, \dots, z^p\right)$ such that $\sum_{0 \le k \le p} z^k \bar{z}^k = 1$, and S' be the set of all points $z' = \left(z'^0, \dots, z'^q\right)$ such that $\sum_{0 \le j \le q} z'^j \bar{z}'^j = 1$. We define

$$V_{kj} = \left\{(z, z') \in S \times S' : z^k z'^j \neq 0,\ 0 \le k \le p,\ 0 \le j \le q\right\}.$$

Then the sets V_{kj} form an open covering of $S \times S'$. Let τ be a complex number such that $\mathrm{Im}(\tau) \neq 0$. In V_{kj} we introduce the following complex coordinates

$$\begin{aligned} {}_k w^h &= \frac{z^h}{z^k},\ {}_j w'^\ell = \frac{z'^\ell}{z'^j},\ h \neq k,\ \ell \neq j,\ 0 \le h \le p,\ 0 \le \ell \le q, \\ t_{kj} &= \frac{1}{2\pi i}\left(\log z^k + \tau \log z'^j\right), \end{aligned} \tag{1.5}$$

where t_{kj} is defined mod 1 and τ. Thus t_{kj} defines a point on the torus $T_{(1,\tau)}$ which is the quotient of $\mathbb{C}$ by the translations 1 and τ. In this way we have $p + q + 1$ coordinates in V_{kj} and these define a map $V_{kj} \longrightarrow \mathbb{C}_{p+q} \times T_{(1,\tau)}$. We show that this map is a homeomorphism. It suffices to show that ${}_kw^h, {}_jw'^\ell$ and t_{kj} determine the z's and z''s uniquely. Now

$$\sum_{h\neq k} {}_kw^h\,{}_k\overline{w}^h = \sum_h \frac{z^h\overline{z}^h}{z^k\overline{z}^k} - 1 = \frac{1}{\left|z^k\right|^2} - 1,$$

so $\left|z^k\right|$ is determined. Similarly, $\left|z'^j\right|$ is determined. By the second equation of (1.5) we have

$$t_{kj} = \frac{1}{2\pi i}\left(\log\left|z^k\right| + \tau\log\left|z'^j\right| + i\arg z^k + \tau i\arg z'^j\right), \quad \operatorname{mod}(1,\tau).$$

Hence $\arg z^k$, $\arg z'^j$ are determined mod 2π. The other z's and z''s are then determined by the first equations of (1.5). This proves that our map is a homeomorphism. p. 6

In $V_{kj}\cap V_{rs}$ we have

$${}_rw^h = \frac{{}_kw^h}{{}_kw^r}, \quad {}_sw'^\ell = \frac{{}_jw'^\ell}{{}_jw'^s}$$

and

$$t_{rs} = t_{kj} + \frac{1}{2\pi i}\left(\log {}_kw^r + \tau\log {}_jw'^s\right), \quad \operatorname{mod}(1,\tau)$$

where we set ${}_kw^k = 1$ and $jw'^j = 1$. Hence we have defined a complex structure on $S\times S'$ with the ${}_kw^h, {}_jw'^\ell, t_{kj}$ as local coordinates in V_{kj}.

2. Complex and Hermitian structures on a vector space

Let V be a real vector space of dimension n. V is said to have a complex structure if there exists a linear endomorphism $J : V \longrightarrow V$, such that $J^2 = -1$, where 1 denotes the identity endomorphism. An eigenvalue of J is a complex number λ such that the equation $Jx = \lambda x$ has a nonzero solution $x \in V$. Applying J to this equation, we get

$$-x = J^2x = \lambda Jx = \lambda^2 x.$$

It follows that $\lambda^2 = -1$ or $\lambda = \pm i$. Since the complex eigenvalues occur in conjugate pairs, it follows that V must be of even dimension $n = 2m$. The following relations are immediately verified:

$$(J-i)(J+i) = (J+i)(J-i) = 0. \tag{2.1}$$

Let V^* be the dual space of V, i.e., the space of all real-valued linear functions over V. We denote the pairing of V and V^* by $\langle x, y^*\rangle$, $x\in V$, $y^*\in V^*$, so that this function is $\mathbb{R}$-linear in each of the arguments. Alternatively, we also write $y^*(x) = \langle x, y^*\rangle$. In addition to V^* we consider $V^*\otimes\mathbb{C}$, i.e., the space of all complex-valued $\mathbb{R}$-linear functions over V. Then $V^*\otimes\mathbb{C}$ is a complex vector space of complex dimension n. An element $f\in V^*\otimes\mathbb{C}$ is said to be of type $(1,0)$ (respectively $(0,1)$) if p. 7

$$f(Jx) = if(x) \ (\text{resp.}\ f(Jx) = -if(x)), \quad x\in V. \tag{2.2}$$

Let $e^{*\alpha}$, $1 \leq \alpha \leq n$, be a basis of V^*. Consider the functions

$$\lambda^\alpha(x) = \langle (J+i)x, e^{*\alpha}\rangle = \langle Jx, e^{*\alpha}\rangle + i\,\langle x, e^{*\alpha}\rangle\,. \tag{2.3}$$

Since $-i$ is an eigenvalue of J of multiplicity m, exactly m of these functions are linearly independent with respect to $\mathbb{C}$. It can be immediately verified that $\lambda^\alpha(x)$ are of type $(1,0)$, and their complex conjugates

$$\overline{\lambda}^\alpha(x) = \langle (J-i)x, e^{*\alpha}\rangle \tag{2.3a}$$

are of type $(0,1)$.

Suppose our basis $e^{*\alpha}$ is so chosen that $\lambda^k(x)$, $1 \leq k \leq m$, are linearly independent with respect to $\mathbb{C}$. We split them into the real and imaginary parts:

$$\lambda^k(x) = \lambda'^k(x) + i\lambda''^k(x). \tag{2.4}$$

We wish to show that $\lambda'^k(x)$, $\lambda''^k(x)$, $1 \leq k \leq m$, are linearly independent with respect to $\mathbb{R}$. In fact, suppose that

$$\sum_k r_k\lambda'^k(x) + \sum_k s_k\lambda''^k(x) = 0, \qquad x \in V,$$

where $r_k, s_k \in \mathbb{R}$. This relation can be written as p. 8

$$\sum_k (r_k - is_k)\,\lambda^k(x) + \sum_k (r_k + is_k)\,\overline{\lambda}^k(x) = 0.$$

Replacing x by Jx and using the fact that $\lambda^k(x)$ and $\overline{\lambda}^k(x)$ are of types $(1,0)$ and $(0,1)$ respectively, we get

$$\sum_k (r_k - is_k)\,\lambda^k(x) - \sum_k (r_k + is_k)\,\overline{\lambda}^k(x) = 0.$$

Adding these two equations, we find

$$\sum (r_k - is_k)\,\lambda^k(x) = 0,$$

which gives $r_k - is_k = 0$, and hence $r_k = s_k = 0$, since $\lambda^k(x)$ are linearly independent over $\mathbb{C}$.

Using the exterior algebra $\bigwedge(V^* \otimes \mathbb{C})$, we can express the fact proved above by

$$\left(\frac{i}{2}\right)^m \wedge_k \lambda^k\overline{\lambda}^k = \bigwedge_k \lambda'^k\lambda''^k \neq 0. \tag{2.5}$$

It follows from (2.5) that $\lambda^k, \overline{\lambda}^k$ are linearly independent over $\mathbb{C}$ and that $V^* \otimes \mathbb{C}$ is a direct sum of $V_{\mathbb{C}} \oplus \overline{V}_{\mathbb{C}}$, where $V_{\mathbb{C}}$ (reps. $\overline{V}_{\mathbb{C}}$) is the space of all elements of $V^* \otimes \mathbb{C}$ of type $(1,0)$ (resp. $(0,1)$). Conversely, a direct sum decomposition of $V^* \otimes \mathbb{C}$ over $\mathbb{C}$ into two subspaces, complex conjugate to each other, defines a complex structure on V, if the subspaces are defined to be consisting of the elements of types $(1,0)$ and $(0,1)$ respectively. This follows from the fact that when x is given the equations in (2.2) determine the values of the elements of $V^* \otimes \mathbb{C}$ at Jx, whereby Jx is determined.

As an example, let e_k, e_{m+k} be a dual basis of λ'^k, λ''^k, so that

$$\lambda^k(e_h) = \lambda'^k(e_{m+h}) = \delta^k_h, \quad 1 \leq h,\ k \leq m, \tag{2.6}$$

all other pairings being zero. This can be written

$$\lambda^k(e_h) = \frac{1}{i}\lambda^k(e_{m+h}) = \delta^k_h. \tag{2.6a}$$

On the other hand, we have p. 9

$$\lambda^k(Jx) = i\lambda^k(x) = -\lambda''^k(x) + i\lambda'^k(x),$$

from which it follows that

$$\lambda^k(Je_h) = \frac{1}{i}\lambda^k(Je_{m+h}) = i\delta^k_h. \tag{2.7}$$

Comparing (2.6a) and (2.7), we get

$$Je_h = e_{m+h}, \quad Je_{m+h} = -e_h. \tag{2.8}$$

The elements λ^k form a basis of $V_{\mathbb{C}}$ over $\mathbb{C}$. Under a change of basis the real-valued $2m$-form in (2.5) will be multiplied by a positive factor. Hence the complex structure J in V defines an orientation of V.

If J defines a complex structure in V, $-J$ does too. The two complex structures are said to be *conjugate* to each other. A form of type $(1,0)$ (resp. type $(0,1)$) in the structure J is a form of type $(0,1)$ (resp. $(1,0)$) in the structure $-J$ and vice versa.

Suppose V is provided with a complex structure J. An hermitian structure in V is a complex-valued function $H(x,y)$, $x, y \in V$, which satisfies the following conditions:

(1) $H(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 H(x_1, y) + \lambda_2 H(x_2, y), x_1, x_2, y \in V, \lambda_1, \lambda_2 \in \mathbb{R}$;
(2) $\overline{H(x,y)} = H(y,x)$;
(3) $H(Jx,y) = iH(x,y)$.

In view of (2), (3) is equivalent to the following:

(3') $H(x,Jy) = -iH(x,y)$.

We split $H(x,y)$ into its real and imaginary parts: p. 10

$$H(x,y) = F(x,y) + iG(x,y). \tag{2.9}$$

Then 2 is equivalent to

$$F(x,y) = F(y,x), \quad G(x,y) = -G(y,x), \tag{2.10}$$

and 3 is equivalent to

$$F(x,y) = G(Jx,y), \quad \text{or } G(x,y) = -F(Jx,y). \tag{2.11}$$

Thus $H(x,y)$ defines a pair of real-valued bilinear functions, of which one is symmetric and the other antisymmetric, which are related to each other by (2.11). Either one of these functions, together with J, determines $H(x,y)$.

The hermitian scalar product $H(x,y)$ is called *positive definite* if the corresponding real-valued symmetric bilinear function $F(x,y)$ is positive definite.

It is known that the space $\bigwedge^2(V^*)$ of exterior forms of degree two is isomorphic to the space of all antisymmetric bilinear functions over V. The isomorphism is established by the fact that it is a vector space isomorphism and that for $\xi, \eta \in V^*$ the bilinear function corresponding to $\xi \wedge \eta$ is

$$(\xi \wedge \eta)(x, y) = \frac{1}{2}\left\{\xi(x)\eta(y) - \xi(y)\eta(x)\right\}, \quad x, y \in V. \tag{2.12}$$

By means of this isomorphism there is an exterior form $\widehat{H}$ of degree two corresponding to the function $-\frac{1}{2}G(x, y)$. $\widehat{H}$ is called the *Kähler form* of the hermitian structure.

We wish to express $H(x, y)$ in terms of the basis λ^K of $V_{\mathbb{C}}$. For this purpose let

$$x = \sum_{\alpha} x^\alpha e_\alpha,\ y = \sum_{\beta} y^\beta e_\beta, \quad 1 \le \alpha, \beta \le 2m,\ 1 \le k, j \le m. \tag{2.13}$$

Then we have p. 11

$$\begin{aligned}
H(x, y) &= H\left(\sum_k \left(x^k e_k + x^{m+k} e_{m+k}\right), y\right) = H\left(\sum_k \left(x^k e_k + x^{m+k} J e_k\right), y\right) \\
&= \sum_k \left(x^k + i x^{m+k}\right) H(e_k, y) \\
&= \sum_{k,j} \left(x^k + i x^{m+k}\right)\left(y^j - i y^{m+j}\right) H(e_k, e_j) \\
&= \sum_{k,j} \lambda^k(x)\overline{\lambda}^j(y) H(e_k, e_j).
\end{aligned}$$

It follows that we can write

$$H = \sum_{k,j} h_{kj} \lambda^k \otimes \overline{\lambda}^j \tag{2.14}$$

where

$$h_{kj} = H(e_k, e_j) = \overline{h}_{jk}. \tag{2.15}$$

To find the expression for the Kähler form $\widehat{H}$, we derive from (2.9)

$$\begin{aligned}
-\frac{1}{2}(G(x, y)) &= \frac{i}{4}\left\{H(x, y) - \overline{H}(x, y)\right\} \\
&= \frac{i}{4}\sum_{k,j} h_{kj}\left\{\lambda^k(x)\overline{\lambda}^j(y) - \overline{\lambda}^j(x)\lambda^k(y)\right\}.
\end{aligned}$$

By (2.12) it follows that

$$\widehat{H} = \frac{i}{2}\sum_{k,j} h_{kj} \lambda^k \wedge \overline{\lambda}^j. \tag{2.16}$$

If a real vector space has a complex structure and in addition to it an hermitian structure, the exterior algebra has rich properties. In particular, a complex-valued exterior form, i.e., an element of the exterior algebra $\bigwedge(V^* \otimes \mathbb{C})$, is said to be of type (p, q), if it is a sum of terms each of which contains p factors λ^k and q factors $\overline{\lambda}^h$. A form of degree r can be written uniquely as a sum

$$\alpha = \sum_{p+q=r} \alpha_{pq}, \quad (p, q) \text{ mutually distinct,} \tag{2.17}$$

where α_{pq} is of type (p, q). The latter will also be denoted by p. 12

$$\alpha_{pq} = \Pi_{pq}\alpha, \tag{2.18}$$

whereby the operators Π_{pq} are defined.

Another operator, which we will denote by L, is defined by

$$L\alpha = \widehat{H} \wedge \alpha. \tag{2.19}$$

L is a real operator in the sense that it maps a real-valued form into a real-valued form. This operator L plays an important role in Hodge's work on transcendental methods in algebraic geometry.

3. Almost complex manifolds; integrability conditions

Let M be a C^∞ manifold of dimension n. To a point $x \in M$ we will denote by T_x and T_x^* the tangent and cotangent spaces respectively. An *almost complex structure* on M is a C^∞ field of endomorphisms $J_x : T_x \to T_x$, such that $J_x^2 = -1_x$, where 1_x denotes the identity endomorphism in T_x.

A manifold which is given an almost complex structure is called *almost complex*. Not all manifolds have this property. In fact, from the discussions in §2 follows the:

Theorem 3(A). An almost complex manifold is even-dimensional and orientable.

Remark. This condition is not sufficient for a manifold to have an almost complex structure. For instance, it was proved by Ehresmann and Hopf that the 4-sphere S^4 cannot be given an almost complex structure [11, p. 217].

Alternatively, an almost complex structure can be defined by the space A of its complex-valued C^∞ forms of type $(1, 0)$. If $\overline{A}$ denotes the space consisting of forms which are conjugate complex to those of A, then at every $x \in M$ we have the direct sum decomposition p. 13

$$T_x^* \otimes \mathbb{C} = A_x \oplus \overline{A}_x, \tag{3.1}$$

where A_x (resp. $\overline{A}_x$) is the space of the forms of A (resp. $\overline{A}$) at x.

To establish the relation between the definitions let x^α, $1 \le \alpha, \beta \le n$, be a local coordinate system. Then a basis in the tangent space T_x is given by $\frac{\partial}{\partial x^\alpha}$ and its dual basis T_x^* consists of the differential forms $\mathrm{d}x^\beta$. The endomorphism J_x will be defined by

$$J_x\left(\sum_\alpha \xi^\alpha \frac{\partial}{\partial x^\alpha}\right) = \sum_{\alpha,\beta} a^\alpha_\beta \xi^\beta \frac{\partial}{\partial x^\alpha}. \tag{3.2}$$

The condition that $J_x^2 = -1_x$ is expressed by

$$\sum_\beta a^\alpha_\beta a^\beta_\gamma = -\delta^\alpha_\gamma, \quad 1 \le \alpha, \beta, \gamma \le n. \tag{3.3}$$

At each point $x \in M$ the discussions of §2 and we see that the forms

$$\sum_\beta \left(a^\alpha_\beta + i\delta^\alpha_\beta\right) \mathrm{d}x^\beta \tag{3.4}$$

are of type $(1, 0)$. They are n in number and exactly $m = \frac{n}{2}$ of them are linearly independent over the ring of complex-valued C^∞ functions (cf. (2.3)). (The situation being local, we restrict ourselves to a sufficiently small neighbourhood. As all our functions are C^∞ unless otherwise specified, we will later on frequently omit the adjective "C^∞".)

Proposition 3(B). A complex manifold has an almost complex structure.

In fact, the complex-valued 1-forms which, in terms of the local coordinates p. 14
z^k, $1 \le k \le m$, are linear combinations of $\mathrm{d}z^k$, are well-defined in a complex manifold M. These we define to be the forms of type $(1, 0)$. Since

$$\left(\frac{i}{2}\right)^m \wedge_k \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^k \neq 0,$$

we have defined an almost complex structure on M.

To describe J in terms of the local coordinates z^k let

$$z^k = x^k + iy^k. \tag{3.5}$$

Then we have, using the fact that $\mathrm{d}z^k$ is of type $(1, 0)$,

$$(\mathrm{d}z^k)\left(\frac{\partial}{\partial x^j}\right) = \delta^k_j, \quad (\mathrm{d}z^k)\left(\frac{\partial}{\partial y^j}\right) = i\delta^k_j,$$

$$(\mathrm{d}z^k)\left(J\frac{\partial}{\partial x^j}\right) = i\delta^k_j, \quad (\mathrm{d}z^k)\left(J\frac{\partial}{\partial y^j}\right) = -\delta^k_j, \quad 1 \le j, k \le m.$$

It follows that

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j}, \quad J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}. \tag{3.6}$$

The question arises whether this is the only way to get an almost complex manifold, i.e., whether every almost complex manifold is complex. This is the case for $n = 2$, but not in general. The question is whether local coordinates x^k, y^k, $1 \le k \le m = \frac{n}{2}$, can be introduced such that, if z^k are defined by (3.5), the forms of type $(1,0)$ are linear combinations of $\mathrm{d}z^k$. Suppose the almost complex structure is locally defined by the forms θ^k of type $(1,0)$ which are linearly independent (over the ring of complex-valued C^∞ functions). Their exterior derivatives can be written p. 15

$$\mathrm{d}\theta^k = \frac{1}{2}\sum_{j,\ell} A^k_{j\ell}\theta^j \wedge \theta^\ell + \sum_{j,\ell} B^k_{j\ell}\theta^j \wedge \overline{\theta}^\ell + \frac{1}{2}\sum_{j,\ell} C^k_{j\ell}\overline{\theta}^j \wedge \overline{\theta}^\ell \tag{3.7}$$

where $A^k_{j\ell}, B^k_{j\ell}, C^k_{j\ell}$ are complex-valued functions satisfying

$$A^k_{j\ell} + A^k_{\ell j} = C^k_{j\ell} + C^k_{\ell j} = 0, \quad 1 \le j, k, \ell \le m. \tag{3.8}$$

The condition

$$\mathrm{d}\theta^k \equiv 0, \quad \mod \theta^j \tag{3.9}$$

remains invariant under a linear transformation of the θ^k. It is satisfied if $\theta^k = \mathrm{d}z^k$. Thus it is a necessary condition for an almost complex structure to arise from a complex structure. We will call (3.9) the *integrability condition*. By (3.7) it can also be written

$$C^k_{j\ell} = 0. \tag{3.9a}$$

Before proceeding, we will express the integrability condition in terms of the tensor field a^α_β which defines the endomorphism J_x. Suppose that our Greek indices range from 1 to n:

$$1 \le \alpha, \beta, \gamma, \lambda, \mu, \rho, \sigma \le n. \tag{3.10}$$

Then we have:

Proposition 3(C) (Eckmann–Frölicher)**.** Let

$$\begin{aligned} a^\alpha_{\beta\gamma} &= -a^\alpha_{\gamma\beta} = \frac{\partial a^\alpha_\beta}{\partial x^\gamma} - \frac{\partial a^\alpha_\gamma}{\partial x^\beta}, \\ t^\alpha_{\beta\gamma} &= \sum_\rho \left(a^\alpha_{\beta\rho}a^\rho_\gamma - a^\alpha_{\gamma\rho}a^\rho_\beta\right). \end{aligned} \tag{3.11}$$

The integrability condition of the almost complex structure defined by the tensor field a^α_β is $t^\alpha_{\beta\gamma} = 0$.

Since the forms of type $(1,0)$ are linear combinations of those in (3.4), the integrability condition can be expressed by p. 16

$$\sum_\beta \mathrm{d}a^\alpha_\beta \wedge dx^\beta \equiv 0, \quad \mod \sum_\lambda \left(a^\gamma_\lambda + i\delta^\gamma_\lambda\right)\mathrm{d}x^\lambda,$$

or

$$\sum_{\beta,\gamma} a^\alpha_{\beta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \equiv 0, \quad \mod \sum_\lambda \left(a^\gamma_\lambda + i\delta^\gamma_\lambda\right)\mathrm{d}x^\lambda.$$

If we equate to zero the forms in (3.4), a fundamental system of solutions of the resulting linear equations in dx^β can be selected from $a^\gamma_\lambda - i\delta^\lambda_\gamma$ (cf. (2.1)). The condition above can therefore be written

$$\sum_{\beta,\gamma} a^\alpha_{\beta\gamma}\left(a^\beta_\lambda - i\delta^\beta_\lambda\right)\left(a^\gamma_\mu - i\delta^\gamma_\mu\right) = 0.$$

Equating to zero either the real or the imaginary part of this equation, we get $t^\alpha_{\beta\gamma} = 0$.

Remark. It can be verified that $t^\alpha_{\beta\gamma}$ are the components of a tensor field.

The integrability condition is identically satisfied when $n = 2$, as can be seen from (3.9a). For $n \geq 4$ the condition is clearly nontrivial. An almost complex structure satisfying the integrability condition is called *integrable*, otherwise *nonintegrable*. An almost complex manifold of dimension ≥ 4 always has a nonintegrable almost complex structure, for even if the given one is integrable, it can be perturbed slightly to give a nonintegrable one.

A significant example of an almost complex manifold is the 6-sphere S^6. From the theory of Lie groups it is known that S^6 can be considered as a coset space $G_2/\mathrm{SU}(3)$, where G_2 is the exceptional simple Lie group of 14 dimensions and $\mathrm{SU}(3)$ is the special unitary group in three variables. From the definition of G_2 and its structure equations one sees immediately that S^6 has a nonintegrable almost complex structure. p. 17

Suppose that we have an integrable almost complex structure. The condition (3.9) suggests us to apply the theorem of Frobenius on completely integrable differential systems. Since the forms are complex-valued, it will be necessary to suppose that the almost complex structure is real analytic, i.e., that the functions a^α_β are real analytic. Under this hypothesis it follows from Frobenius's theorem that there exist complex local coordinates z^k such that the forms of type $(1,0)$ are linear combinations of dz^k. In a neighbourhood where two such local coordinate systems z^k and w^j are both valid dw^j are linear combinations of dz^k, which implies that w^j are holomorphic functions of z^k. Thus the manifold has a complex structure.

This theorem that a complex structure can be introduced in a manifold with an integrable almost complex structure is also true if the latter is C^∞ or satisfies even weaker smoothness conditions. This was first proved by A. Newlander and L. Nirenberg [10]. Subsequent proofs were given by A. Nijenhuis and W. B. Woolf, J. Kohn and L. Hörmander. These proofs are rather difficult. The case $n = 2$ is a classical theorem of Korn and Lichtenstein which asserts that a two-dimensional riemannian metric of class $C^{1,\alpha}$ $(0 < \alpha < 1)$ is locally conformal to a flat metric. Even the proof of the Korn–Lichtenstein theorem is not simple [1].

Thus we see that integrable almost complex structures and complex structures are essentially identical. In some of the problems it is not necessary to make use of the local complex coordinates z^k, and the Newlander–Nirenberg theorem will not be needed. But we will not insist on this point.

The integrability condition (3.9) or $t^{\alpha}_{\beta\gamma} = 0$ (by 3(C)) is a criterion for deciding whether a given almost complex structure is integrable. It gives no information on the problem whether an almost complex manifold can be given a complex structure, whose underlying almost complex structure may be different from the given one. Recently van de Ven gave examples of compact four-dimensional almost complex manifolds which do not have any complex structure; his proof makes use of the Atiyah–Singer index theorem [12]. It is an outstanding problem whether S^6 can have a complex structure. p. 18

Let M be an almost complex manifold of dimension $n = 2m$. All complex-valued C^∞ forms of type (p, q) constitute a module A_{pq} over the ring of complex-valued C^∞ functions. The following properties are easily verified:

(1) If $\alpha \in A_{pq}$, then $\overline{\alpha} \in A_{qp}$.
(2) If $\alpha \in A_{pq}, \beta \in A_{rs}$, then $\alpha \wedge \beta \in A_{p+r,q+s}$.
(3) $\mathrm{d}A_{pq} \subset A_{p+2,q-1} + A_{p+1,q} + A_{p,q+1} + A_{p-1,q+2}$.
(4) $A_{pq} = 0$ if p or $q > m$.

From (3) we define, for $\alpha \in A_{pq}$, the operators

$$\partial\alpha = \Pi_{p+1,q}\mathrm{d}\alpha, \quad \overline{\partial}\alpha = \Pi_{p,q+1}\mathrm{d}\alpha. \tag{3.12}$$

If the almost complex structure is integrable, (3) becomes

$$\mathrm{d}A_{pq} \subset A_{p+1,q} + A_{p,q+1}, \tag{3I}$$

as follows immediately from (3.7). We can then write

$$\mathrm{d} = \partial + \overline{\partial}. \tag{3.13}$$

Since $\mathrm{d}^2 = 0$, we get

$$\partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0.$$

Equating to zero the terms of different types, we find

$$\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0. \tag{3.14}$$

The last condition gives rise to the following form of the integrability condition: p. 19

Proposition 3(D). An almost complex structure is integrable if and only if

$$\overline{\partial}^2 = 0.$$

It remains to prove that the integrability condition is satisfied if $\overline{\partial}^2 = 0$. In fact, let F be a complex-valued C^∞ function. We write

$$\mathrm{d}F = \sum_k F_k\theta^k + \sum_k G_k\overline{\theta}^k.$$

Then we have

$$\partial F = \sum_k F_k\theta^k, \quad \overline{\partial}F = \sum_k G_k\overline{\theta}^k,$$

and

$$\begin{aligned}\overline{\partial}^2 F &= \Pi_{0,2}\mathrm{d}\overline{\partial}F = \Pi_{0,2}\mathrm{d}(\overline{\partial} - \mathrm{d})F = -\Pi_{0,2}\mathrm{d}\partial F \\ &= -\sum_{j,k,\ell} F_k C^k_{j\ell}\overline{\theta}^j \wedge \overline{\theta}^\ell.\end{aligned}$$

Since this expression is zero for any F, we get $C^k_{j\ell} = 0$, which is the integrability condition (3.9a).

From now on suppose M is a complex manifold. A form $\alpha \in A_{pq}$ is called $\overline{\partial}$-closed if $\overline{\partial}\alpha = 0$. Let C_{pq} be the space of $\overline{\partial}$-closed forms of type (p, q). The quotient groups

(3.15) $$D_{pq}(M) = C_{pq}/\overline{\partial}A_{p,q-1}$$

are called the *Dolbeault groups* of M.

The Dolbeault groups are analogous to the de Rham groups of a real manifold, whose definitions we recall as follows: Let A_r be the space of real-valued C^∞ forms of degree r, and C_r be the subspace of the forms of A_r which are annihilated by d. Then the de Rham groups are p. 20

(3.16) $$R_r(M) = C_r/\mathrm{d}A_{r-1}.$$

Both the de Rham groups and the Dolbeault groups are isomorphic to cohomology groups with coefficient sheaves, which will be treated in §4. Before concluding this section, we will prove an important lemma:

Lemma 3(E) (Dolbeault–Grothendieck)**.** In the number space $\mathbb{C}_m$ with the coordinates z^k, $1 \le k \le m$, let D be the polydisc $|z^k| < r^k$, and let D' be the smaller polydisc $|z^k| < r'^k$, $r'^k < r^k$. Let α be a form of type (p, q), $q \ge 1$, in D such that $\overline{\partial}\alpha = 0$. There exists a form β of type $(p, q-1)$ in D such that $\overline{\partial}\beta = \alpha$ in D'.

We consider first a special case of this lemma, i.e., $m = 1$, $(p, q) = (0, 1)$. We write z for z^1. Then

$$\alpha = f(z)\mathrm{d}\bar{z},$$

where $f(z)$ is a complex-valued C^∞ function. The form β sought is a function which satisfies the partial differential equation

(3.17) $$\frac{\partial\beta}{\partial\bar{z}} = f(z),$$

where

(3.18) $$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad z = x + iy.$$

We note that if the equation (3.17) is split into its real and imaginary parts we get an elliptic system of two equations of the first order in two independent and two dependent variables.

Let $z, \zeta \in D$ and regard z to be fixed. We have the relation

$$\mathrm{d}\left(\frac{\beta d\zeta}{\zeta - z}\right) = \beta_{\bar{\zeta}}\frac{\mathrm{d}\bar{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z}.$$

Suppose $z \in D'$ and let Δ_ε be a disc of radius ε about z, ε being sufficiently small. Applying Stokes' theorem to the domain $D' - \Delta_\varepsilon$, we get p. 21

$$\int_{\partial D'} \frac{\beta(\zeta)\mathrm{d}\zeta}{\zeta - z} - \int_{\partial \Delta_\varepsilon} \frac{\beta(\zeta)\mathrm{d}\zeta}{\zeta - z} = \int_{D' - \Delta_\varepsilon} \frac{\beta_{\bar\zeta}\mathrm{d}\bar\zeta \wedge \mathrm{d}\zeta}{\zeta - z}.$$

The second integral at the left-hand side tends to $2\pi i \beta(z)$ as $\varepsilon \longrightarrow 0$. We have therefore the generalised Cauchy integral formula

$$2\pi i \beta(z) = \int_{\partial D'} \frac{\beta \mathrm{d}\zeta}{\zeta - z} + \int_{D'} \frac{\beta_{\bar\zeta}\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta}{\zeta - z}. \tag{3.19}$$

Taking the conjugate complex of this equation and replacing $\bar\beta$ by β, we have also

$$-2\pi i \beta(z) = \int_{\partial D'} \frac{\beta \mathrm{d}\bar\zeta}{\bar\zeta - \bar z} - \int_{D'} \frac{\beta_{\zeta}\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta}{\bar\zeta - \bar z}. \tag{3.19a}$$

Equation (3.19) shows that if (3.17) has a solution $\beta(z)$, it is given by

$$2\pi i \beta(z) = \int_{D'} \frac{f(\zeta)\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta}{\zeta - z} + g(z) \tag{3.20}$$

where $g(z)$ is a holomorphic function. It remains to verify that the function in (3.20) satisfies the equation (3.17).

For this purpose we consider the relation

$$\mathrm{d}\left\{f(\zeta) \log|\zeta - z|^2 \mathrm{d}\bar\zeta\right\} = f_\zeta \log|\zeta - z|^2 \mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta + \frac{f}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta,$$

and apply Stokes' theorem to the domain $D' - \Delta_\varepsilon$. As $\varepsilon \longrightarrow 0$, the integral

$$\int_{\partial \Delta_\varepsilon} f(\zeta) \log|\zeta - z|^2 \mathrm{d}\bar\zeta$$

tends to zero, because, if $|f(\zeta)| \leq B$, we have p. 22

$$\left| \int_{\partial \Delta_\varepsilon} f(\zeta) \log|\zeta - z|^2 \mathrm{d}\bar\zeta \right| \leq 4\pi B \varepsilon \log \varepsilon.$$

We have therefore

$$\begin{aligned} \int_{\partial D'} f(\zeta) \log|\zeta - z|^2 \mathrm{d}\bar\zeta - \int_{D'} f_\zeta \log|\zeta - z|^2 \mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta \\ = \int_{D'} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta = 2\pi i \beta(z) - g(z), \end{aligned}$$

by (3.20). Differentiating under the integral sign with respect to $\bar z$, we get

$$-\int_{\partial D'} \frac{f(\zeta)}{\bar\zeta - \bar z}\mathrm{d}\bar\zeta + \int_{D'} f_\zeta \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta}{\bar\zeta - \bar z} = 2\pi i \frac{\partial \beta}{\partial \bar z}.$$

This differentiation can be justified, essentially because the resulting integrals exist. By (3.19a) (with β replaced by f) we see that the function $\beta(z)$ in (3.20) satisfies the equation (3.17).

It is important to remark that the proof shows that if the function $f(z)$ is holomorphic in some complex parameters, the same is true for the solution β.

To prove the general case we introduce the hypothesis (H_j) : α does not contain $d\bar{z}^{j+1}, \ldots, d\bar{z}^m$. We shall prove that if the lemma is true with the additional hypothesis (H_{j-1}), it is true with the additional hypothesis (H_j). Under the hypothesis (H_0), we have $\alpha = 0$, and the lemma is true. On the other hand, the hypothesis (H_m) is empty. Thus the above induction statement will imply the lemma.

Suppose therefore that the lemma is true with the additional hypothesis (H_{j-1}). If α does not involve $d\bar{z}^{j+1}, \ldots, d\bar{z}^m$, we write

$$\alpha = \left(d\bar{z}^j \wedge \lambda\right) + \mu,$$

where λ and μ are forms of types $(p, q-1)$ and (p, q) respectively and do not p. 23
contain $d\bar{z}^j, \ldots, d\bar{z}^m$. Since $\bar{\partial}\alpha = 0$, their coefficients are holomorphic in $z^{j+1}, \ldots, z^m$. By the special case proved above, we can find a form λ' of type $(p, q-1)$ which satisfies the equation

$$\frac{\partial}{\partial \bar{z}^j}\lambda' = \lambda$$

in D' and whose coefficients are holomorphic in $z^{j+1}, \ldots, z^m$; here the operator $\frac{\partial}{\partial \bar{z}^j}$ means the operator applied to each of the coefficients. Then $\bar{\partial}\lambda' - d\bar{z}^j \wedge \lambda = \nu$ (say) does not contain $d\bar{z}^j, \ldots, d\bar{z}^m$, and

$$\alpha = \bar{\partial}\lambda' + \mu - \nu.$$

Since $\bar{\partial}\alpha = 0$, we have $\bar{\partial}(\mu - \nu) = 0$. But $\mu - \nu$ does not contain $d\bar{z}^j, \ldots, d\bar{z}^m$, so that, by our induction hypothesis we can find a form ρ of type $(p, q-1)$ in D satisfying

$$\mu - \nu = \bar{\partial}\rho \quad \text{in } D'.$$

Thus $\alpha = \bar{\partial}(\lambda' + \rho)$ and the induction is complete.

4. Sheaves and cohomology

Sheaf theory is a basic tool in the study of complex manifolds. We will review its main ideas and the cohomology theory built on it. For details cf. [5] or [3].

Let M be a topological space. A sheaf of abelian groups is a topological space $\mathcal{S}$ together with mapping $\pi : \mathcal{S} \to M$, such that the following conditions are satisfied:

(1) π is a local homeomorphism;
(2) for each point $x \in M$ the set $\pi^{-1}(x)$ (called the stalk over x) has the structure of an abelian group;
(3) the group operations are continuous in the topology of $\mathcal{S}$.

Let U be an open set of M. A section of the sheaf $\mathcal{S}$ over U is a continuous p. 24
mapping $f : U \to \mathcal{S}$, such that $\pi \circ f =$ identity. The set $\Gamma(U, \mathcal{S})$ of all the sections over U forms an abelian group, for if $f, g \in \Gamma(U, \mathcal{S})$, we can define $f - g$

by $(f-g)(x) = f(x) - g(x)$, $x \in U$. The zero of the group $\Gamma(U, \mathcal{S})$ is the zero section which assigns the zero of the stalk $\pi^{-1}(x)$ to every $x \in U$.

If V is an open subset of U, there is a homomorphism $\rho_{VU} : \Gamma(U, \mathcal{S}) \to \Gamma(V, \mathcal{S})$ defined by restriction. These conditions motivate the following definition:

A presheaf of abelian groups over M consists of:

(1) a basis for the open sets of M;
(2) an abelian group $\mathcal{S}_U$ assigned to each open set U of the basis; and
(3) a homomorphism $\rho_{VU} : \mathcal{S}_U \to \mathcal{S}_V$ associated to each inclusion $V \subset U$, such that $\rho_{WV}\rho_{VU} = \rho_{WU}$ whenever $W \subset V \subset U$.

From the presheaf one can construct the sheaf by a limit process.

Suppose now that M is a complex manifold. The following sheaves will play an important role in future discussions:

(1) the sheaf $\mathcal{A}_{pq}$ of germs of complex-valued C^∞ forms of type (p, q). In particular, we will write $\mathcal{A} = \mathcal{A}_{00}$, the sheaf of germs of complex-valued C^∞ functions.
(2) the sheaf $\mathcal{C}_{pq}$ of germs of complex valued C^∞ forms of type (p, q), which are closed under $\bar\partial$. We write $\mathcal{O} = \mathcal{C}_{00}$, the sheaf of germs of holomorphic functions. For complex manifolds this is the most important sheaf.
(3) the sheaf $\mathcal{O}^*$ of germs of holomorphic functions which vanish nowhere. Here the group operation is the multiplication of germs of holomorphic functions. p. 25

A section of the sheaf $\mathcal{A}_{pq}$ is a form of type (p, q), etc. Thus, in the notation of §3,

$$A_{pq} = \Gamma(M, \mathcal{A}_{pq}), \quad C_{pq} = \Gamma(M, \mathcal{C}_{pq}), \quad \text{etc.} \tag{4.1}$$

Let

$$\pi : \mathcal{S} \to M, \quad \tau : \mathcal{T} \to M$$

be two sheaves of abelian groups over the same space M. A sheaf mapping $\phi : \mathcal{S} \to \mathcal{T}$ is a continuous mapping such that $\pi = \tau \circ \phi$, i.e., a mapping which preserves the stalks: $\phi(\pi^{-1}(x)) \subset \tau^{-1}(x)$. ϕ is called a *sheaf homomorphism* if its restriction to every stalk is a homomorphism of the groups.

If $\mathcal{Q} \to M$ is a third sheaf over M, the sequence of sheaves

$$0 \to \mathcal{S} \xrightarrow{i} \mathcal{T} \xrightarrow{\phi} \mathcal{Q} \to 0 \tag{4.2}$$

connected by homomorphisms is called an *exact sequence* if at each stage the kernel of one homomorphism is identical to the image of the preceding homomorphism. We describe this by saying that $\mathcal{S}$ is a *subsheaf* of $\mathcal{T}$ and $\mathcal{Q}$ is the *quotient sheaf* of $\mathcal{T}$ by $\mathcal{S}$.

It follows from the Dolbeault–Grothendieck lemma proved in §3 that the sequence

$$0 \to \mathcal{C}_{pq} \xrightarrow{i} \mathcal{A}_{pq} \xrightarrow{\bar\partial} \mathcal{C}_{p,q+1} \to 0 \tag{4.3}$$

is exact. Here i is the inclusion homomorphism and $\overline{\partial}$ is the homomorphism on sheaves induced by the $\overline{\partial}$-operator. The Dolbeault–Grothendieck lemma says that $\overline{\partial}$ is onto; the exactness of the sequence at the other stages is obvious.

To develop the cohomology theory with a coefficient sheaf we suppose that M is a paracompact Hausdorff space. Let $\mathcal{U} = \{U_i\}$ be a locally finite open covering of M. The nerve $N(\mathcal{U})$ of the covering $\mathcal{U}$ is a simplicial complex whose p. 26
vertices are the members U_i of the covering such that $U_{i_0}, U_{i_1}, \dots, U_{i_q}$ span a q-dimensional simplex if and only if the intersection $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q} \neq \emptyset$. Let $\pi : \mathcal{S} \to M$ be a sheaf of abelian groups over M. A q-cochain of $N(\mathcal{U})$ with coefficients in the sheaf $\mathcal{S}$ is a function f which associates to each q-simplex $\sigma = U_{i_0} U_{i_1} \cdots U_{i_q} \in N(\mathcal{U})$ a section $f(\sigma) \in \Gamma(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{S})$. Since the set of sections is an abelian group, the set of all q-cochains form an abelian group $C^q(N(\mathcal{U}), \mathcal{S})$.

A coboundary operator

$$\delta_q : C^q(N(\mathcal{U}), \mathcal{S}) \to C^{q+1}(N(\mathcal{U}), \mathcal{S})$$

is defined as follows: if $f \in C^q(N(\mathcal{U}), \mathcal{S})$ and $\sigma = U_0 \cdots U_{q+1}$, then $\delta_q f \in C^{q+1}(N(\mathcal{U}), \mathcal{S})$ has for σ the value

$$(\delta_q f)(\sigma) = \sum_{j=0}^{q+1} (-1)^j \rho_0 f(U_0 \cdots U_{j-1} U_{j+1} \cdots U_{q+1}), \tag{4.4}$$

where ρ_0 denotes the restriction of the sections to the open set $U_0 \cap \cdots \cap U_{q+1}$.

It is immediately verified that

$$\delta_{q+1} \delta_q = 0, \quad q \geq 0. \tag{4.5}$$

The kernel of δ_q is called the group of all q-cocycles and will be denoted by $Z^q(N(\mathcal{U}), \mathcal{S})$. The image of δ_{q-1} is called the group of all q-coboundaries and will be denoted by $B^q(N(\mathcal{U}), \mathcal{S})$. As a consequence of (4.5), a q-coboundary is a q-cocycle, and the quotient group

$$H^q(N(\mathcal{U}), \mathcal{S}) = Z^q(N(\mathcal{U}), \mathcal{S}) / B^q(N(\mathcal{U}), \mathcal{S}), \quad B^0 = 0, \tag{4.6}$$

is called the *q-th cohomology group of the nerve $N(U)$ with the coefficient sheaf $\mathcal{S}$*.

The zeroth cohomology group has the simple interpretation: p. 27

$$H^0(N(\mathcal{U}), \mathcal{S}) = \Gamma(M, \mathcal{S}). \tag{4.7}$$

By a standard process initiated by Čech, one can pass from the cohomology groups $H^q(N(\mathcal{U}), \mathcal{S})$ relative to all the locally finite open coverings $\mathcal{U}$ of M, to the cohomology groups $H^q(M, \mathcal{S})$, $q \geq 0$, of the space M itself.

Let $\pi : \mathcal{S} \to M$ be a sheaf of abelian groups over M and let $\mathcal{U} = \{U_i\}$ be a locally finite open covering of M. A *partition of unity* of the sheaf $\mathcal{S}$ subordinate to the covering $\mathcal{U}$ is a collection of sheaf homomorphisms $\eta_i : \mathcal{S} \to \mathcal{S}$ with the properties:

(1) η_i is the zero map in an open neighbourhood of $M - U_i$;
(2) $\sum_i \eta_i = 1$, the latter being the identity mapping of the sheaf $\mathcal{S}$.

A sheaf $\mathcal{S}$ of abelian groups is *fine* if it admits a partition of unity subordinate to any locally finite open covering.

Examples of fine sheaves are $\mathcal{A}_{pq}$. Examples of sheaves which are generally not fine include:

(1) the constant sheaf;
(2) the sheaf $\mathcal{C}_{pq}$.

Fine sheaves play a catalytic role in the cohomology theory of sheaves, because of the theorem:

If $\mathcal{S}$ is fine, then $H^q(M, \mathcal{S}) = 0, q \geq 1$.

A sheaf homomorphism $i : \mathcal{S} \to \mathcal{T}$ induces a homomorphism $\Gamma(U, \mathcal{S}) \to \Gamma(U, \mathcal{T})$ for every open set U of M, and hence a homomorphism

$$i^q : C^q(N(\mathcal{U}), \mathcal{S}) \to C^q(N(\mathcal{U}), \mathcal{T}).$$

This leads to an induced homomorphism

$$i^q : H^q(M, \mathcal{S}) \to H^q(M, \mathcal{T}), \quad q \geq 0.$$

As a result of the exact sequence (4.2) we wish to describe a homomorphism p. 28

$$\delta^q : H^q(M, \mathcal{Q}) \to H^{q+1}(M, \mathcal{S})$$

and to connect the homomorphisms into a long exact sequence. The exact sequence (4.2) induces the exact sequence

$$0 \to C^q(N(\mathcal{U}), \mathcal{S}) \xrightarrow{i^q} C^q(N(\mathcal{U}), \mathcal{T}) \xrightarrow{\phi^q} C^q(N(\mathcal{U}), \mathcal{Q}).$$

We put

$$\overline{C}^q(N(\mathcal{U}), \mathcal{Q}) = \phi^q C^q(N(\mathcal{U}), \mathcal{T}) \subset C^q(N(\mathcal{U}), \mathcal{Q}), \tag{4.8a}$$

so that the sequence

$$0 \to C^q(N(\mathcal{U}), \mathcal{S}) \xrightarrow{i^q} C^q(N(\mathcal{U}), \mathcal{T}) \xrightarrow{\phi^q} \overline{C}^q(N(\mathcal{U}), \mathcal{Q}) \to 0$$

is exact. Let

$$\begin{aligned} \overline{Z}^q(N(\mathcal{U}), \mathcal{Q}) &= \left\{ f \in \overline{C}^q(N(\mathcal{U}), \mathcal{Q}) : \delta_q f = 0 \right\}, \\ \overline{H}^q(N(\mathcal{U}), \mathcal{Q}) &= \overline{Z}^q(N(\mathcal{U}), \mathcal{Q}) / \delta_{q-1} \overline{C}^{q-1}(N(\mathcal{U}), \mathcal{Q}). \end{aligned} \tag{4.8b}$$

Consider the diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C^q(N(\mathcal{U}), \mathcal{S}) \arrow[r, "i^q"] \arrow[d, "\delta^q"'] & C^q(N(\mathcal{U}), \mathcal{T}) \arrow[r, "\phi^q"] \arrow[d, "\delta^q"'] & \overline{C}^q(N(\mathcal{U}), \mathcal{Q}) \arrow[r] \arrow[d, "\delta^q"'] & 0 \\
0 \arrow[r] & C^{q+1}(N(\mathcal{U}), \mathcal{S}) \arrow[r, "i^{q+1}"] \arrow[d, "\delta^{q+1}"'] & C^{q+1}(N(\mathcal{U}), \mathcal{T}) \arrow[r, "\phi^{q+1}"] \arrow[d, "\delta^{q+1}"'] & \overline{C}^{q+1}(N(\mathcal{U}), \mathcal{Q}) \arrow[r] \arrow[d, "\delta^{q+1}"'] & 0 \\
0 \arrow[r] & C^{q+2}(N(\mathcal{U}), \mathcal{S}) \arrow[r, "i^{q+2}"] \arrow[d] & C^{q+2}(N(\mathcal{U}), \mathcal{T}) \arrow[r, "\phi^{q+2}"] \arrow[d] & \overline{C}^{q+2}(N(\mathcal{U}), \mathcal{Q}) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

This diagram is commutative, in the sense that the image of a cochain depends only on its final position and is independent of the paths taken. Moreover, the horizontal sequences are exact. To an element of $\overline{H}^q(M, \mathcal{Q})$ we take a representative q-cocycle, i.e., an element $u \in \overline{C}^q(N(\mathcal{U}), \mathcal{Q})$, such that $\delta^q u = 0$. There ex- p. 29
ists $v \in C^q(N(\mathcal{U}), \mathcal{T})$, such that $\phi^q v = u$. Then $\phi^{q+1} \circ \delta^q v = \delta^q \circ \phi^q v = \delta^q u = 0$, and there exists $w \in C^{q+1}(N(\mathcal{U}), \mathcal{S})$, satisfying $i^{q+1} w = \delta^q v$. w is a cocycle, for

$$i^{q+2} \circ \delta^{q+1} w = \delta^{q+1} \circ i^{q+1} w = \delta^{q+1} \circ \delta^q v = 0,$$

so that $\delta^{q+1} w = 0$. By further "chasing" of the diagram, it can be shown that the element of $H^{q+1}(N(\mathcal{U}), \mathcal{S})$ defined by w is independent of the various choices made. This defines a homomorphism

$$\delta^q : \overline{H}^q(N(\mathcal{U}), \mathcal{Q}) \to H^{q+1}(N(\mathcal{U}), \mathcal{S}).$$

This definition is valid for a general topological space M. It can be proved that if M is Hausdorff and paracompact, then

$$\overline{H}^q(M, \mathcal{Q}) \cong H^q(M, \mathcal{Q}).$$

A *fundamental fact* in cohomology theory is the result: If the sequence of sheaves (4.2) is exact, the sequence of cohomology groups

(4.9)

$$\begin{aligned} 0 \longrightarrow H^0(M, \mathcal{S}) \xrightarrow{i^0} H^0(M, \mathcal{T}) \xrightarrow{\phi^0} H^0(M, \mathcal{Q}) \xrightarrow{\delta^0} H^1(M, \mathcal{S}) \\ \xrightarrow{i^1} H^1(M, \mathcal{T}) \xrightarrow{\phi^1} H^1(M, \mathcal{Q}) \xrightarrow{\delta^1} H^2(M, \mathcal{S}) \longrightarrow \cdots \end{aligned}$$

is exact.

We apply this result to the exact sequence (4.3). A section of the induced sequence of cohomology groups will be

(4.10)

$$\begin{aligned} \cdots \longrightarrow H^{r-1}(M, \mathcal{A}_{pq}) \longrightarrow H^{r-1}(M, \mathcal{C}_{p,q+1}) \longrightarrow H^r(M, \mathcal{C}_{pq}) \\ \longrightarrow H^r(M, \mathcal{A}_{pq}) \longrightarrow \cdots \end{aligned}$$

Since the sheaf $\mathcal{A}_{pq}$ is fine, we have

$$H^r(M, \mathcal{A}_{pq}) = 0, \quad r \geq 1,$$

and it follows from the exactness of (4.10) that we have the isomorphisms p. 30

$$\begin{aligned} H^r(M, \mathcal{C}_{pq}) &\cong H^{r-1}(M, \mathcal{C}_{p,q+1}) \cong \cdots \cong H^1(M, \mathcal{C}_{p,q+r-1}) \\ &\cong H^0(M, \mathcal{C}_{p,q+r}) \Big/ \overline{\partial} H^0(M, \mathcal{A}_{p,q+r-1}) . \end{aligned} \tag{4.11}$$

Comparing with (4.1), we see that the latter is the Dolbeault group $D_{p,q+r}(M)$. By changing notation, we get

$$D_{pq}(M) \cong H^q(M, \mathcal{C}_{p0}). \tag{4.12}$$

This gives a sheaf-theoretic interpretation of the Dolbeault groups. Notice that $\mathcal{C}_{p0}$ is the sheaf of germs of forms of type $(p, 0)$ with holomorphic coefficients, and, in particular, $\mathcal{C}_{0,0} = \mathcal{O}$.

The sequences (4.3) can be combined into one sequence

$$0 \to \mathcal{C}_{p0} \xrightarrow{i} \mathcal{A}_{p0} \xrightarrow{\overline{\partial}} \mathcal{A}_{p1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}_{pq} \to \cdots, \tag{4.13}$$

where i is inclusion and $\overline{\partial}$ is defined by the $\overline{\partial}$-operator. The Dolbeault–Grothendieck lemma says that the sequence (4.13) is exact; the subsheaf of $\mathcal{A}_{pq}$ which is the image of the preceding homomorphism and the kernel of the next one is precisely $\mathcal{C}_{pq}$. Since $\mathcal{A}_{pq}$ is fine, (4.13) is called a *fine resolution of the sheaf* $\mathcal{C}_{p0}$.

A similar, but simpler, situation prevails in the case of a real differentiable manifold M. Let $\mathcal{A}^r$ be the sheaf of germs of C^∞ real-valued differential forms of degree r, and let $\mathcal{C}^r$ be the subsheaf of $\mathcal{A}^r$ consisting of germs of closed r-forms. Then the sequence

$$0 \to \mathbb{R} \xrightarrow{i} \mathcal{A}^0 \xrightarrow{\mathrm{d}} \mathcal{A}^1 \to \cdots \xrightarrow{\mathrm{d}} \mathcal{A}^r \to \cdots, \tag{4.14}$$

where $\mathbb{R}$ is the constant sheaf of real numbers and i is inclusion, is exact. (4.14) is a fine resolution of the sheaf $\mathbb{R}$. From the exactness of (4.14) follows the de Rham isomorphism p. 31

$$R_r(M) \cong H^r(M;\mathbb{R}), \tag{4.15}$$

where the left-hand side is the r-dimensional de Rham group of M (cf. (3.16)).

The sheaf theory discussed above can be extended to other algebraic structures, such as sheaf of rings, sheaf of modules, etc. Moreover, the group operation on a stalk may not be abelian, in which case, however, there will not be a cohomology theory.

5. Complex vector bundles; connections

Throughout this section we will denote by M a C^∞ differentiable manifold, and we will develop the properties of complex vector bundles over M. For economy the adjective "complex" is sometimes omitted.

Let

$$F = \mathbb{C}_q = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{q}$$

be the complex vector space of complex dimension q. Suppose F is acted on to the right by $\mathrm{GL}(q;\mathbb{C})$, the general linear group in q complex variables, so that $\xi \cdot g \in F$ and

$$(\xi g)h = \xi(gh), \quad \xi \in F,\ g, h \in \mathrm{GL}(q;\mathbb{C}). \tag{5.1}$$

A complex vector bundle E over M consists of a space E and a projection

$$\psi : E \to M, \tag{5.2}$$

such that the following conditions are fulfilled:

(1) Every point $x \in M$ has a neighbourhood U for which there exists a homeomorphism (a "chart") p. 32

$$\phi_U : U \times F \to \psi^{-1}(U), \tag{5.3}$$

with

$$\psi \circ \phi_U(y, \xi) = y, \quad y \in U,\ \xi \in F. \tag{5.4}$$

(2) In the intersection $U \cap V$ of two such neighbourhoods U, V there exists a C^∞ map $g_{UV} : U \cap V \to \mathrm{GL}(q; \mathbb{C})$, such that

$$\phi_U(x, \xi) = \phi_V(x, \xi'), \quad x \in U \cap V;\ \xi, \xi' \in F, \tag{5.5}$$

if and only if

$$\xi g_{UV}(x) = \xi'. \tag{5.6}$$

These functions g_{UV}, the so-called *transition functions*, satisfy the compatibility relations

$$\begin{cases} g_{UV}^{-1} = g_{VU}, \\ g_{UV} g_{VW} g_{WU} = 1 \quad \text{in } U \cap V \cap W. \end{cases} \tag{5.7}$$

If $q = 1$, the vector bundle is called a *line bundle*. The set $\psi^{-1}(x)$, $x \in M$, is a complex vector space of dimension q, and is called the fibre at x. Our assumptions are such that the complex linear structures on the fibres have a meaning.

As a consequence of this remark, operations on complex vector spaces which commute with the actions of the general linear groups can be extended to operations on bundles. Among the most important operations are:

(1) The dual bundle E^* of E. Its transition functions are ${}^t g_{UV}^{-1}$ (i.e., the transpose inverse of g_{UV}, when the latter is interpreted as a nonsingular $(q \times q)$-matrix).

(2) If E' and E'' are two complex vector bundles over M with the transition functions g'_{UV}, g''_{UV} respectively, their *direct sum* or *Whitney sum* $E' \oplus E''$ is defined by the transition functions p. 33

$$\begin{pmatrix} g'_{UV} & 0 \\ 0 & g''_{UV} \end{pmatrix}.$$

Similarly, their tensor product $E' \otimes E''$ is defined by the transition functions $g'_{UV} \otimes g''_{UV}$. If the dimensions of the fibres of E', E'' are q', q'' respectively, the fibre dimension of $E' \oplus E''$ is $q' + q''$ and that of $E' \otimes E''$ is $q'q''$.

(3) The bundle $\mathrm{Hom}(E', E'') \cong E'^* \otimes E''$.

In order that the notion of a vector bundle be meaningful, it is desirable to introduce an equivalence relation which amounts to a change of the charts. Let E and E' be two vector bundles over M with the same fibre dimension q which, relative to an open covering $\{U, V, \ldots\}$ of M, are given by the charts ϕ_U, ϕ'_U and the transition functions g_{UV}, g'_{UV} respectively. They are called *equivalent* if to each U there is a C^∞ map $g_U : U \to \mathrm{GL}(q;\mathbb{C})$, such that

$$\phi_U(x, \xi g_U) = \phi'_U(x, \xi), \quad x \in U, \ \xi \in F. \tag{5.8}$$

In terms of the transition functions condition (5.8) implies:

$$g'_{UV} = g_U g_{UV} g_V^{-1}. \tag{5.9}$$

An immediate question is the scope of the equivalence classes of complex vector bundles over M, or, more specifically, whether there exist bundles which are (globally) not products of M with F. For $q = 1$ the answer is given by the:

Theorem 5(A). All the C^∞ complex line bundles over a differentiable manifold M form a group which is isomorphic to $H^2(M, \mathbb{Z})$, the second cohomology group of M with integer coefficients.

To prove this theorem let $\mathcal{A}$ be the sheaf of germs of complex-valued C^∞ func- p. 34
tions and let $\mathcal{A}^*$ be the sheaf of germs of nowhere zero complex-valued C^∞ functions, the latter with multiplication as the group operation. By the compatibility relations (5.7) and by (5.9) it follows that the equivalence classes of C^∞ complex line bundles are in one-one correspondence with the elements of the cohomology group $H^1(M, \mathcal{A}^*)$. Thus all the line bundles of M form a group, and the multiplication of two line bundles is given by the tensor product. From now on we will not distinguish between a line bundle and an equivalence class of line bundles.

Consider the sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{A} \xrightarrow{e} \mathcal{A}^* \to 0, \tag{5.10}$$

where i is inclusion and e is defined by

$$e(f(x)) = \exp\big(2\pi i f(x)\big), \quad f(x) \in \mathcal{A}. \tag{5.11}$$

The sequence (5.10) is obviously an exact sequence. From its exactness follows the exactness of the following sequence of cohomology groups:

$$H^1(M, \mathcal{A}) \xrightarrow{e^1} H^1(M, \mathcal{A}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \xrightarrow{i^2} H^2(M, \mathcal{A}).$$

Since $\mathcal{A}$ is a fine sheaf, the groups at both ends of this sequence are zero, and we get the isomorphism stated in the theorem.

If $E \in H^1(M, \mathcal{A}^*)$ is a complex line bundle, $\delta E \in H^2(M, \mathbb{Z})$ is called its *Chern class*.

The simple conclusion in 5(A) is possible, because the group $\mathrm{GL}(1;\mathbb{C})$ is abelian. For general q there are Chern classes

$$c_i(E) \in H^{2i}(M, \mathbb{Z}), \quad 1 \le i \le q,$$

which are the simplest invariants of a complex vector bundle, but we will postpone their discussion to a later section. p. 35

Let E be a complex vector bundle over M, and let T^* be the cotangent bundle of M. Denote by $\Gamma(E)$ and $\Gamma(T^* \otimes E)$ respectively the spaces of sections of E and of the tensor product $T^* \otimes E$ (over $\mathbb{C}$). A *connection* on E is an operator

$$D : \Gamma(E) \to \Gamma(T^* \otimes E) \tag{5.12}$$

which satisfies the conditions:

$$\begin{aligned} D(\gamma_1 + \gamma_2) &= D\gamma_1 + D\gamma_2, \quad \gamma_1, \gamma_2 \in \Gamma(E), \\ D(f\gamma) &= \mathrm{d}f \cdot \gamma + fD\gamma, \quad \gamma \in \Gamma(E), \end{aligned} \tag{5.13}$$

where $f \in A$ (= the space of complex-valued C^∞ functions over M) and $\mathrm{d}f \cdot \gamma = \mathrm{d}f \otimes \gamma$, the tensor product here being over A.

We will first study the local properties of a connection. Let U be an open set of M, and let $s_1, \dots, s_q$ be a *frame field* over U, i.e., q sections of the bundle E over U, such that $s_1(x), \dots, s_q(x)$, $x \in U$, are linearly independent. Then we can write

$$Ds_i = \sum_j \omega_i^j s_j, \quad 1 \le i, j \le q, \tag{5.14}$$

where ω_i^j are complex-valued 1-forms in U. For economy of writing we will express (5.14) in matrix form. In fact, let

$${}^t s = (s_1, \dots, s_q), \quad \omega = (\omega_i^j), \tag{5.15}$$

so that s itself is a one-columned matrix of q sections. Then (5.14) can be written

$$Ds = \omega s. \tag{5.16}$$

The matrix ω completely determines the connection. In fact, any section ξ of E over U can be written

$$\xi = \sum_i \xi^i s_i, \tag{5.17}$$

where ξ^i are complex-valued C^∞ functions in U. By (5.13), we have p. 36

$$D\xi = \sum_i \left(\mathrm{d}\xi^i + \sum_j \xi^j \omega_j^i \right) s_i. \tag{5.18}$$

We call ω the *connection matrix*.

The section ξ is called *horizontal* if

$$D\xi = 0 \tag{5.19}$$

or

$$\mathrm{d}\xi^i + \sum_j \xi^j \omega_j^i = 0. \tag{5.19a}$$

Equations (5.19a) are a system of total differential equations and generally do not have a solution. However, when restricted to a parametrised curve C with parameter t, they become a system of ordinary differential equations, and a solution $\xi^i(t)$ is determined by its initial values $\xi^i(t_0)$ at a given point $t = t_0$. The mapping $C \to \psi^{-1}(C)$ defined by assigning to the point $t \in C$ the vector $\gamma = \sum_i \xi^i(t)s_i$ is called a *lifting* of the curve C to the bundle E, and it is called a *horizontal lifting* if γ satisfies (5.19) or (5.19a). In classical language a lifting is called a *vector field* along C and a horizontal lifting is called a *parallel vector field* along C.

Let

$$s' = gs \tag{5.20}$$

be a new frame field, where g is a nonsingular $(q \times q)$-matrix of complex-valued C^∞ functions in U. By (5.13), we have

$$Ds' = \omega' s', \tag{5.21}$$

where

$$\omega' g = \mathrm{d}g + g\omega. \tag{5.22}$$

This is an important formula, giving the effect on the connection matrix under a change of the frame field. p. 37

By taking the exterior derivative of (5.22) and using (5.22), we get

$$\Omega' g = g\Omega, \tag{5.23}$$

where

$$\Omega = \mathrm{d}\omega - \omega \wedge \omega, \tag{5.24}$$

and Ω' is defined similarly in terms of the connection matrix ω'. Ω is a $(q \times q)$-matrix of exterior 2-forms, and is called the *curvature matrix* relative to the frame field s.

The simple transformation law (5.23) implies the following: The vanishing of Ω is a condition independent of the choice of s. A connection satisfying $\Omega = 0$ is called *flat*.

Exterior differentiation of (5.24) gives

$$\mathrm{d}\Omega + \Omega \wedge \omega - \omega \wedge \Omega = 0, \tag{5.25}$$

which is called the *Bianchi identity*.

The example of the curvature matrix motivates the definition: Suppose there is associated to every frame field s a $(q \times q)$-matrix ϕ_s of forms of degree k, such that under a change of the frame field (5.20) we have

$$\phi_{s'} = g\phi_s g^{-1}. \tag{5.26}$$

Such a collection of matrices $\{\phi_s\}$ is called a *tensorial matrix of the adjoint type*. (The name arises from the adjoint representation of the group $GL(q;\mathbb{C})$.) By taking the exterior derivative of (5.26) and using (5.22), we get

$$D\phi_{s'} = gD\phi_s g^{-1}, \tag{5.27}$$

where p. 38

$$D\phi_s = \mathrm{d}\phi_s - \omega \wedge \phi_s + (-1)^k \phi_s \wedge \omega \tag{5.28}$$

and $D\phi_{s'}$ is defined similarly with the connection matrix ω' relative to the frame field s'. Thus $D\phi_s$ is a tensorial matrix of $(k+1)$-forms of the adjoint type. It is called the *covariant differential* of ϕ_s.

The covariant differential of the curvature matrix will not lead to anything significant because the Bianchi identity (5.25) can be written

$$D\Omega = 0. \tag{5.29}$$

Here and later we will frequently omit the subscript s, when the frame field is fixed through the discussion.

By (5.28) it can be immediately verified that

$$D^2\phi = DD\phi = [\phi, \Omega], \tag{5.30}$$

where the "commutator" is defined by

$$[\phi, \Omega] = \phi \wedge \Omega - \Omega \wedge \phi. \tag{5.31}$$

We now consider a complex-valued function $P(A_1, \dots, A_r)$, whose arguments are the $(q \times q)$-matrices A_i, $1 \le i \le r$, and which is $\mathbb{C}$-linear in each of the arguments. In fact, if

$$A_i = (a_{i,\alpha\beta}), \quad 1 \le i \le r,\ 1 \le \alpha, \beta \le q, \tag{5.32}$$

then

$$P(A_1, \dots, A_r) = \sum \lambda_{\alpha_1 \dots \alpha_r \beta_1 \dots \beta_r} a_{1\alpha_1\beta_1} \dots a_{r\alpha_r\beta_r}, \tag{5.33}$$

where the λ's are complex numbers and the summation is over the α's and the β's from 1 to q. Such a function (or polynomial) is called *invariant*, if

$$P(gA_1g^{-1}, \dots, gA_rg^{-1}) = P(A_1, \dots, A_r) \tag{5.34}$$

for every nonsingular matrix g. It will be called *symmetric*, if its value remains unchanged on a permutation of its arguments. p. 39

Examples of symmetric invariant polynomials can be obtained as follows: Let A be a $(q \times q)$-matrix, I be the $(q \times q)$-unit matrix, and let

$$\det\left(I + \frac{i}{2\pi}A\right) = \sum_{0 \le j \le q} \binom{q}{j} P_j(A), \tag{5.35}$$

where $P_j(A)$ is a polynomial of degree j in the elements of A. Let $P_j(A_1, \dots, A_j)$ be the completely polarised polynomial of $P_j(A)$, so normalised that

$$P_j(A, \dots, A) = P_j(A). \tag{5.36}$$

From the definition (5.35), we have

$$P_j(gAg^{-1}) = P_j(A). \tag{5.37}$$

Since $P_j(A_1, \dots, A_j)$ can be expressed in terms of $P_j(A)$ for different arguments A, for instance,

$$P_2(A_1, A_2) = \frac{1}{2}\left\{P_2(A_1 + A_2) - P_2(A_1) - P_2(A_2)\right\},$$

it follows that $P_j(A_1, \dots, A_j)$ are invariant.

Suppose $P(A_1, \dots, A_r)$ is an invariant polynomial, so that the equation (5.34) is fulfilled. Let

$$g = I + g'.$$

Then

$$g^{-1} = I - g' + \cdots,$$

where the dots involve terms containing higher powers of the elements of g'. Substituting into (5.34) and retaining only the terms which are linear in the elements of g', we get

$$\sum_{1 \le i \le r} P(A_1, \dots, g'A_i - A_i g', \dots, A_r) = 0, \tag{5.38}$$

for any matrix g'. This identity remains true, when $A_1, \dots, A_r$ are matrices whose elements are differential forms (in which case P is a complex-valued form). p. 40

Suppose the elements of A_i are forms of degree d_i. It follows from (5.38) that

$$\begin{aligned} &\sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_{i-1}} P(A_1, \dots, \theta \wedge A_i, \dots, A_r) \\ &+ \sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_i + 1} P(A_1, \dots, A_i \wedge \theta, \dots, A_r) = 0, \end{aligned} \tag{5.39}$$

where θ is a $(q \times q)$-matrix of 1-forms. In fact, θ is a sum of matrices of the form $g'\alpha$, where g' is a matrix of functions and α is a one-form. Since (5.39) is linear in θ, it suffices to prove it for the case $\theta = g'\alpha$. By moving α to the front of the expressions, we see that (5.39) for the case $\theta = g'\alpha$ follows immediately from (5.38).

The invariant polynomials constitute a link between the local properties of a connection and its global properties. In fact, we say that a family of matrices $\{\phi_s\}$ is a tensorial matrix of k-forms of the adjoint type in M, if such a matrix ϕ_s is associated to every local frame field s such that the relation (5.26) holds under a change of the frame field (5.20). If $P(A_1, \dots, A_r)$ is an invariant polynomial and A_i is a tensorial matrix of the adjoint type in M, whose elements are forms of degree d_i, $1 \le i \le r$, then $P(A_1, \dots, A_r)$ is a form of degree $d_1 + \cdots + d_r$ which is *globally defined* in M. Moreover, it follows from (5.27) and (5.39) that its exterior derivative is

$$\mathrm{d}P(A_1, \dots, A_r) = \sum_{1 \le i \le r} (-1)^{d_1 + \cdots + d_{i-1}} P(A_1, \dots, DA_i, \dots, A_r). \tag{5.40}$$

For the polynomials $P_j(A)$ defined in (5.35) we have therefore p. 41

$$\mathrm{d}P_j(\Omega) = 0, \tag{5.41}$$

because of the Bianchi identity (5.29). Thus $P_j(\Omega)$ is a closed form of degree $2j$ in M and defines an element of the de Rham group $R_{2j} \cong H^{2j}(M, \mathbb{C})$ with complex coefficients.

Theorem 5(B). Let $\psi : E \to M$ be a complex vector bundle with fibre dimension q. Let Ω be the curvature matrix of a connection in the bundle. Then a change of the connection modifies $P_j(\Omega)$, $1 \leq j \leq q$, by an additive term of the form $\mathrm{d}Q$, where Q is a form of degree $2j-1$ in M.

The following proof of 5(B) is due to Weil. Let ω, Ω and $\widetilde{\omega}$, $\widetilde{\Omega}$ be respectively the connection and curvature matrices of two connections relative to the same frame field s. If s and s' are related by (5.20), we have (5.22) and the corresponding relation

$$\widetilde{\omega}' g = dg + g\widetilde{\omega},$$

for the second connection. Putting

$$\eta = \widetilde{\omega} - \omega, \quad \eta' = \widetilde{\omega}' - \omega', \tag{5.42}$$

we get

$$\eta' g = g\eta. \tag{5.43}$$

Thus, η, the difference of two connection matrices, is a tensorial matrix of 1-forms of the adjoint type. We put

$$\omega_t = \omega + t\eta, \quad 0 \leq t \leq 1. \tag{5.44}$$

Then ω_t is a connection matrix depending on the parameter t, which reduces to ω and $\widetilde{\omega}$ for $t = 0$ and $t = 1$ respectively.

The curvature matrix of the connection ω_t is by definition p. 42

$$\Omega_t = \mathrm{d}\omega_t - \omega_t \wedge \omega_t = \Omega + tD\eta - t^2 \eta \wedge \eta, \tag{5.45}$$

where the covariant differential is taken with respect to the connection ω.

Let $P(A_1, \ldots, A_r)$ be a symmetric invariant polynomial. Let

$$\begin{aligned} P(A) &= P(A, \ldots, A), \\ Q(B, A) &= rP(B, \underbrace{A, \ldots, A}_{r-1}). \end{aligned} \tag{5.46}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\Omega_t) = Q(D\eta, \Omega_t) - 2tQ(\eta \wedge \eta, \Omega_t).$$

On the other hand, we have, from (5.45) and (5.30),

$$\begin{aligned} D\Omega_t = tD^2\eta + t^2[\eta, D\eta] &= t[\eta, \Omega] + t^2[\eta, D\eta] \\ &= t[\eta, \Omega_t], \end{aligned}$$

so that

$$\begin{aligned} \mathrm{d}Q(\eta, \Omega_t) &= Q(D\eta, \Omega_t) - r(r-1)P(\eta, D\Omega_t, \Omega_t, \ldots, \Omega_t) \\ &= Q(D\eta, \Omega_t) - r(r-1)tP(\eta, [\eta, \Omega_t], \Omega_t, \ldots, \Omega_t). \end{aligned}$$

Equation (5.39) gives, with $\theta = A_1 = \eta, A_2 = \cdots = A_r = \Omega_t$,

$$2Q(\eta \wedge \eta, \Omega_t) - r(r-1)P(\eta, [\eta, \Omega_t], \Omega_t, \ldots, \Omega_t) = 0.$$

Combining the last two equations, we get

$$\mathrm{d}Q(\eta, \Omega_t) = Q(D\eta, \Omega_t) - 2tQ(\eta \wedge \eta, \Omega_t).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\Omega_t) = \mathrm{d}Q(\eta, \Omega_t). \tag{5.47}$$

Integrating with respect to t, we get

$$P(\widetilde{\Omega}) - P(\Omega) = \mathrm{d} \int_0^1 Q(\eta, \Omega_t) \mathrm{d}t. \tag{5.48}$$

This proves 5(B). p. 43

Special cases of $P_j(\Omega)$ are:

$$\begin{aligned} P_1(\Omega) &= \frac{i}{2\pi q} \sum_j \Omega^j_j, \\ P_2(\Omega) &= \frac{-2}{(2\pi)^2 q(q-1)} \sum_{j<k} \left(\Omega^j_j \Omega^k_k - \Omega^k_j \Omega^j_k \right), \end{aligned} \tag{5.49}$$

where Ω^k_j, $1 \le j, k \le q$, are the elements of the curvature matrix Ω.

We will now define an hermitian structure on the bundle (5.2). We recall that an hermitian structure on a complex vector space V is a complex-valued function $H(\xi, \eta)$, $\xi, \eta \in V$, such that

$$\begin{aligned} &(1) \quad H(\lambda_1 \xi_1 + \lambda_2 \xi_2, \eta) = \lambda_1 H(\xi_1, \eta) + \lambda_2 H(\xi_2, \eta), \\ &\qquad\qquad \lambda_1, \lambda_2 \in \mathbb{C}, \ \xi_1, \xi_2, \eta \in V, \\ &(2) \quad \overline{H(\xi, \eta)} = H(\eta, \xi). \end{aligned} \tag{5.50}$$

It is called positive definite if

$$H(\xi, \xi) > 0, \quad \xi \ne 0. \tag{5.51}$$

An *hermitian structure* on the complex vector bundle (5.2) is a C^∞ field of positive definite hermitian structures in the fibres of E. That is, if ξ, η are two C^∞ sections of the bundle, $H(\xi, \eta)$ is a complex-valued C^∞ function having properties corresponding to (5.50). A complex vector bundle with an hermitian structure is called an *hermitian vector bundle*. By a partition of unity argument, it can be shown that every complex vector bundle can be given an hermitian structure.

To every frame field s the hermitian structure defines an hermitian matrix p. 44

$$H_s = {}^t\overline{H}_s = \left(H(s_i, s_j) \right), \quad 1 \le i, j \le q, \tag{5.52}$$

and is in turn completely determined by this matrix. Under a change of frame field (5.20) this matrix is transformed according to

$$H_{s'} = g H_s {}^t\overline{g}, \tag{5.53}$$

where

$$H_{s'} = \left(H(s'_i, s'_j) \right), \quad 1 \le i, j \le q.$$

A connection in an hermitian vector bundle is called *admissible*, if $H(\xi, \eta)$ remains constant when ξ, η are horizontal sections along arbitrary curves. Let

$$h_{ik} = H(s_i, s_k), \quad 1 \le i, j, k \le q \tag{5.54}$$

and let

$$\xi = \sum_i \xi^i s_i, \quad \eta = \sum_j \eta^j s_j.$$

Then

$$H(\xi, \eta) = \sum_{i,k} h_{ik} \xi^i \overline{\eta}^k.$$

The sections ξ, η being horizontal, we have (5.19a) and a similar equation for η^k. It follows that

$$\mathrm{d}H(\xi, \eta) = \sum_{i,k} \left(\mathrm{d}h_{ik} - \sum h_{jk} \omega^j_i - \sum h_{ij} \overline{\omega}^j_k \right) \xi^i \overline{\eta}^k.$$

Since horizontal sections along curves exist with arbitrary initial values of ξ^i, η^k, the condition for an admissible connection becomes

(5.55) $$\mathrm{d}h_{ik} - \sum_j h_{jk} \omega^j_i - \sum_j h_{ij} \overline{\omega}^j_k = 0,$$

or, in matrix notation

(5.55a) $$\mathrm{d}H = \omega H + H^t \overline{\omega},$$

where the subscript s is dropped. By an elementary extension argument, it follows from (5.55a) that an admissible connection always exists in an hermitian vector bundle. By taking the exterior derivative of (5.55a), we get p. 45

(5.56) $$\Omega H + H^t \overline{\Omega} = 0,$$

i.e., ΩH is skew-hermitian.

A frame field s of an hermitian vector bundle is called *unitary*, if $H_s = I$ (= the unit matrix). Relative to a unitary frame field, the equations (5.55a) and (5.56) become respectively

(5.57) $$\omega + {}^t\overline{\omega} = 0,$$

(5.58) $$\Omega + {}^t\overline{\Omega} = 0,$$

i.e., the connection and curvature matrices ω and Ω are both skew-hermitian.

It follows from (5.56) that for an hermitian vector bundle with an admissible connection, we have

(5.59) $$\det\left(I + \frac{i}{2\pi}\Omega\right) = \det\left(I - \frac{i}{2\pi}\overline{\Omega}\right) = \overline{\det\left(I + \frac{i}{2\pi}\Omega\right)}.$$

For the coefficients $P_j(A)$ defined in (5.35) and their polarised polynomials $P_j(A_1, \ldots, A_j)$ we write

(5.60) $$P_j(A_1, \ldots, A_j) = (\mathrm{Re}\, P_j)(A_1, \ldots, A_j) + i(\mathrm{Im}\, P_j)(A_1, \ldots, A_j),$$

so that $\mathrm{Re}\, P_j$ and $\mathrm{Im}\, P_j$ are real-valued and $\mathbb{R}$-linear in each of their arguments. Let

$$(\mathrm{Re}\, P_j)(\Omega) = (\mathrm{Re}\, P_j)(\Omega, \ldots, \Omega),$$
$$(\mathrm{Im}\, P_j)(\Omega) = (\mathrm{Im}\, P_j)(\Omega, \ldots, \Omega).$$

Then it follows from (5.59) that

$$(\operatorname{Im} P_j)(\Omega) = 0$$

for the curvature matrix Ω of an admissible connection of an hermitian structure. It follows from Theorem 5(B) that for any connection the element of the de Rham group R_{2j} determined by $(\operatorname{Im} P_j)(\Omega)$ is zero. On the other hand, we will show later that the element determined by $(\operatorname{Re} P_j)(\Omega)$ is what is called the jth Chern class of the bundle E with real or integer coefficients. p. 46

6. Holomorphic vector bundles and line bundles

Let M be a complex manifold of dimension m and let $\psi : E \longrightarrow M$ be a complex vector bundle over M with fibre dimension q. Relative to a covering $\{U, V, \ldots\}$ of M let g_{UV} be the transition functions of E. The bundle is called *holomorphic* if all these functions g_{UV} are *holomorphic* (i.e., g_{UV}, considered as a nonsingular $(q \times q)$-matrix, is a matrix of holomorphic functions in $U \cap V$). If $q = 1$, E is called a *holomorphic line bundle*.

An example of a holomorphic vector bundle over M is the tangent bundle of M. Let $z_U^1, \ldots, z_U^m$ (respectively $z_V^1, \ldots, z_V^m$) be the local coordinates in U (resp. in V). Then the tangent bundle has as transition functions the jacobian matrices

$$j_{UV} = \frac{\partial\left(z_U^1, \ldots, z_U^m\right)}{\partial\left(z_V^1, \ldots, z_V^m\right)}. \tag{6.1}$$

Let E be a holomorphic bundle. A section γ of E over a neighbourhood $U \subset M$ is *holomorphic* if the components of γ relative to a chart are holomorphic functions. A frame field $s = {}^t(s_1, \ldots, s_q)$ is holomorphic if each s_i is a holomorphic section. When s and s' are holomorphic frame fields, the matrix g in the equation (5.20) is a matrix of holomorphic functions. A connection such that the connection matrix is a matrix of 1-forms of type $(1, 0)$ relative to a holomorphic frame field will be called a *connection of type* $(1, 0)$.

Suppose an hermitian structure is defined in E. From (5.55a) it follows that it has a uniquely defined admissible connection of type $(1, 0)$. In fact, its connection matrix is p. 47

$$\omega = \partial H \cdot H^{-1}. \tag{6.2}$$

From (6.2) we find that its curvature matrix is

$$\Omega = -\partial\bar{\partial} H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \bar{\partial} H \cdot H^{-1}, \tag{6.3}$$

so that Ω is of type $(1, 1)$. (A matrix of differential forms is said to be of type (p, q) if each element is a form of type (p, q).)

In case $q = 1$, the matrices in question are (1×1)-matrices:

$$H = (h),\ \Omega = (\Omega), \quad h > 0, \tag{6.4}$$

and (6.3) can be written

$$\Omega = -\partial\bar{\partial} \log h. \tag{6.5}$$

Notice that for $q = 1$ a tensorial matrix of the adjoint type is a form in M, so that Ω is globally defined in M. We call $\frac{i}{2\pi}\Omega$ the *curvature form* of the connection.

On M let $\mathcal{O}$ be the sheaf of germs of holomorphic functions and $\mathcal{O}^*$ be the sheaf of germs of nowhere zero holomorphic functions, the latter with multiplication as the group operation. A meromorphic function is locally the ratio of two holomorphic functions. Let μ be the sheaf of germs of meromorphic functions. Then $\mathcal{O}^*$ is a subsheaf of μ and the quotient sheaf $\mathcal{D}$ is by definition a sheaf of germs of divisors. The latter is locally represented by a meromorphic function defined up to the multiplication by a nowhere zero holomorphic function. We have the exact sequence

$$0 \to \mathcal{O}^* \xrightarrow{i} \mu \xrightarrow{k} \mathcal{D} \to 0. \tag{6.6}$$

Its induced exact cohomology sequence has the part p. 48

(6.7)

$$0 \to H^0(M, \mathcal{O}^*) \xrightarrow{i^0} H^0(M, \mu) \xrightarrow{k^0} H^0(M, \mathcal{D}) \xrightarrow{\delta^0} H^1(M, \mathcal{O}^*) \to \cdots .$$

From the exactness it follows that the quotient group

$$H^0(M, \mathcal{D})/k^0 H^0(M, \mu) \tag{6.8}$$

is isomorphic to a subgroup of $H^1(M, \mathcal{O}^*)$. An element of $H^0(M, \mathcal{D})$ is called a *divisor*. Two divisors are called *linearly equivalent* if they differ from each other (multiplicatively) by a meromorphic function in M. Thus the group (6.8) is the group of divisor-classes with respect to linear equivalence. On the other hand, $H^1(M, \mathcal{O}^*)$ is the group of all holomorphic line bundles over M, the group operation being defined by tensor product. Kodaira and Spencer proved that if M is a nonsingular projective variety the group (6.8) is isomorphic to $H^1(M, \mathcal{O}^*)$, [9].

We wish to study the group $H^1(M, \mathcal{O}^*)$. For this purpose we consider the exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \to 0, \tag{6.9}$$

where e is defined by

$$e(f(x)) = 2\pi i \exp(f(x)), \quad f(x) \in \mathcal{O}. \tag{6.10}$$

The sequence (6.9) leads to the homomorphism

$$\delta^1 : H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}), \tag{6.11}$$

and we wish to describe the image of δ^1.

Let $\mathcal{A}^r_{\mathbb{R}}$ be the sheaf of germs of real-valued C^∞ r-forms in M and $\mathcal{C}^r_{\mathbb{R}}$ be the subsheaf of $\mathcal{A}^r_{\mathbb{R}}$ consisting of those germs which are closed under d, so that $\mathcal{C}^0_{\mathbb{R}}$ is the constant sheaf $\mathbb{R}$. Then we have the exact sequence

$$0 \to \mathcal{C}^r_{\mathbb{R}} \xrightarrow{\ell^r} \mathcal{A}^r_{\mathbb{R}} \xrightarrow{\mathrm{d}} \mathcal{C}^{r+1}_{\mathbb{R}} \to 0, \tag{6.12}$$

whose induced exact cohomology sequence is p. 49

$$\begin{aligned}\cdots \longrightarrow H^p(M,\mathcal{C}^r_{\mathbb{R}}) &\xrightarrow{\varrho^{p,r}} H^p(M,\mathcal{A}^r_{\mathbb{R}}) \xrightarrow{\mathrm{d}} H^p(M,\mathcal{C}^{r+1}_{\mathbb{R}}) \\ &\xrightarrow{\partial^{p,r}} H^{p+1}_{\mathbb{R}}(M,\mathcal{C}^r_{\mathbb{R}}) \qquad \cdots .\end{aligned} \tag{6.13}$$

We note that $\mathcal{A}^r_{\mathbb{R}}$ is a fine sheaf.

We now write the following diagram of cohomology groups of M connected by homomorphisms:

$$\begin{tikzcd} & & H^1(\mathcal{A}^0_{\mathbb{R}}) = 0 \arrow[d, "\mathrm{d}"] & \\ H^0(\mathcal{A}^1_{\mathbb{R}}) \arrow[r, "\mathrm{d}"] & H^0(\mathcal{C}^2_{\mathbb{R}}) \arrow[r, "\partial^{0,1}"] & H^1(\mathcal{C}^1_{\mathbb{R}}) \arrow[r, "\varrho^{1,1}"] \arrow[d, "\partial^{1,0}"] & H^1(\mathcal{A}^1_{\mathbb{R}}) = 0 \\ H^1(\mathcal{O}^*) \arrow[r, "\delta^1"] & H^2(\mathbb{Z}) \arrow[r, "j"] & H^2(\mathbb{R}) \arrow[d, "\varrho^{2,0}"] & \\ & & H^2(\mathcal{A}^0_{\mathbb{R}}) = 0 & \end{tikzcd} \tag{6.14}$$

In this diagram the manifold M is omitted in the notation of the cohomology groups for the sake of simplicity. The vertical sequence and the top horizontal sequence are exact, being parts of the sequence (6.13). The homomorphism j in the second horizontal sequence is induced by the coefficient homomorphism $j : \mathbb{Z} \to \mathbb{R}$. For an hermitian line bundle $E \in H^1(M,\mathcal{O}^*)$ we wish to determine

$$\left(\partial^{0,1}\right)^{-1} \circ \left(\partial^{1,0}\right)^{-1} \circ j \circ \delta^1 E, \tag{6.15}$$

which is a real-valued closed 2-form in M. In fact, we wish to show that the negative of the form (6.15) is the curvature form $\frac{i}{2\pi}\Omega$, up to an additive term $\mathrm{d}\alpha$, where α is a real-valued 1-form in M.

This "diagram chasing" is not difficult, the main point being to remember the definitions of the homomorphisms in question. Before proceeding we will give some more discussion of the hermitian structure of a holomorphic line bundle p. 50
E and its curvature form. In fact, let $\mathcal{U} = \{U, V, W, \ldots\}$ be an open covering of M which is sufficiently fine. Let s_U be a holomorphic frame field over U. The fibre dimension being one, s_U is given by a nowhere zero holomorphic function in U. Let

$$h_U = H(s_U, s_U) > 0. \tag{6.16}$$

The change of frame field in $U \cap V$ is given by

$$s_U g_{UV} = s_V, \tag{6.17}$$

where g_{UV} is a nowhere zero holomorphic function in $U \cap V$. From (6.16) and (6.17) we derive

$$h_U \left|g_{UV}\right|^2 = h_V \quad \text{in } U \cap V. \tag{6.18}$$

It follows that

$$\partial\bar{\partial}\log h_U = \partial\bar{\partial}\log h_V \quad \text{in } U \cap V, \tag{6.19}$$

which gives a verification of the remark following formula (6.5).

Suppose M is equipped with a riemannian metric and that the members of the covering $\mathcal{U}$ are convex. Then the intersection of any number of the members of the covering, if nonempty, is convex. In $U \cap V$ ($\neq \varnothing$) we construct the holomorphic function f_{UV} satisfying

$$g_{UV} = \exp(2\pi i f_{UV}). \tag{6.20}$$

In $U \cap V \cap W \neq \varnothing$ let

$$c_{UVW} = f_{UV} + f_{VW} + f_{WU}. \tag{6.21}$$

Then

$$\exp(2\pi i c_{UVW}) = g_{UV} g_{VW} g_{WU} = 1,$$

so that c_{UVW} is an integer. The two-cochain of the nerve $N(\mathcal{U})$ of the covering $\mathcal{U}$ defined by assigning to the simplex UVW the integer c_{UVW} is a two-cocycle and defines a representative of $\delta^1 E$ and hence of $j \circ \delta^1 E$. p. 51

Next we wish to find a representative of the element of $H^1(\mathcal{C}^1_{\mathbb{R}})$ which is mapped by $\partial^{1,0}$ to $j \circ \delta^1 E$. This will be given by a real-valued closed 1-form in every $U \cap V \neq \varnothing$, and we see that the form $\frac{1}{2}\mathrm{d}\left(f_{UV} + \bar{f}_{UV}\right)$ has the desired property. In fact, by (6.20) and (6.18) we have

$$\begin{aligned}
\frac{1}{2}\mathrm{d}\left(f_{UV} + \bar{f}_{UV}\right) &= \frac{1}{4\pi i}\left\{\partial \log g_{UV} - \bar{\partial}\log \bar{g}_{UV}\right\} \\
&= \frac{1}{4\pi i}(\partial - \bar{\partial})\log\left|g_{UV}\right|^2 \\
&= \frac{1}{4\pi i}(\partial - \bar{\partial})(-\log h_U + \log h_V).
\end{aligned}$$

By the definition of $\partial^{0,1}$ we get a representative of (6.15) as

$$\frac{1}{4\pi i}\mathrm{d}(\partial - \bar{\partial})\log h_U = \frac{i}{2\pi}\partial\bar{\partial}\log h_U = -\frac{i}{2\pi}\Omega.$$

Thus we have the:

Theorem 6(A). Let $E \in H^1(M, \mathcal{O}^*)$ be a holomorphic line bundle over a complex manifold M. Let $c(E) = -\delta^1 E \in H^2(M, \mathbb{Z})$ be its Chern class. Suppose that E has an hermitian structure with the curvature form $\frac{i}{2\pi}\Omega$. Then the element of the de Rham group $R_2(M)$ defined by $\frac{i}{2\pi}\Omega$ corresponds to the element $jc(E) \in H^2(M, \mathbb{R})$ via the de Rham isomorphism (4.15), j being induced by the coefficient homomorphism $j : \mathbb{Z} \to \mathbb{R}$.

Consider the de Rham isomorphism

$$R_2(M) \xrightarrow{\rho} H^2(M, \mathbb{R}). \tag{4.15}$$

There is a subgroup $R_{11}(M)$ of $R_2(M)$, whose elements have as representatives forms of type $(1, 1)$. (Recall that an element γ of $R_2(M)$ is a class of forms $\alpha + \mathrm{d}\beta$, where α is a given real-valued closed 2-form in M and β runs over all real-valued 1-forms in M. Any such form $\alpha + \mathrm{d}\beta$ is called a representative of γ.) Let p. 52

$$\rho R_{11}(M) = H^2_{(1,1)}(M, \mathbb{R}),$$
$$H^2_{(1,1)}(M, \mathbb{Z}) = j^{-1} H^2_{(1,1)}(M, \mathbb{R}). \tag{6.22}$$

Then we have the:

Theorem 6(B). The image of the homomorphism δ^1 in (6.11) is $H^2_{(1,1)}(M, \mathbb{Z})$.

Since the curvature form $\frac{i}{2\pi}\Omega$ is of type $(1, 1)$, we have proved

$$\delta^1 H^1(M, \mathcal{O}^*) \subset H^2_{(1,1)}(M, \mathbb{Z}).$$

To prove inclusion in the other direction, consider the following exact sequence induced by (6.9):

$$\begin{aligned} \cdots \longrightarrow H^1(M, \mathbb{Z}) &\xrightarrow{i^1} H^1(M, \mathcal{O}) \xrightarrow{e^1} H^1(M, \mathcal{O}^*) \\ &\xrightarrow{\delta^1} H^2(M, \mathbb{Z}) \xrightarrow{i^2} H^2(M, \mathcal{O}) \longrightarrow \cdots \end{aligned} \tag{6.23}$$

It suffices to prove that $i^2 H^2_{(1,1)}(M, \mathbb{Z}) = 0$.

As previously let $\mathcal{A}^r$ be the sheaf of germs of complex-valued C^∞ r-forms and $\mathcal{C}^r$ be the subsheaf of the germs of $\mathcal{A}^r$ closed under d. Also let $\mathcal{A}^{pq}$ be the sheaf of germs of C^∞ forms of type (p, q) and $\mathcal{C}^{pq}$ be the subsheaf of germs of $\mathcal{A}^{pq}$ closed under $\overline{\partial}$. Thus by definition $\mathcal{C}^0 = \mathbb{C}$ and $\mathcal{C}^{00} = \mathcal{O}$. We have the diagram p. 53

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^r \arrow[r, "k"] \arrow[d, "\Pi_{0r}"] & \mathcal{A}^r \arrow[r, "\mathrm{d}"] \arrow[d, "\Pi_{0r}"] & \mathcal{C}^{r+1} \arrow[r] \arrow[d, "\Pi_{0,r+1}"] & 0 \\ 0 \arrow[r] & \mathcal{C}^{0r} \arrow[r, "k'"] & \mathcal{A}^{0r} \arrow[r, "\overline{\partial}"] & \mathcal{C}^{0,r+1} \arrow[r] & 0 \end{tikzcd} \tag{6.24}$$

where k and k' are inclusions. This diagram is clearly commutative. Moreover both horizontal sequences are exact. The above diagram implies the following commutative diagram of cohomology groups, where the manifold M is omitted in the notation:

$$\begin{tikzcd} H^0(\mathcal{C}^2)/\mathrm{d}H^0(\mathcal{A}^1) \arrow[r, "\Delta^0"] \arrow[d, "\Pi_{02}"] & H^1(\mathcal{C}^1) \arrow[r, "\Delta^1"] \arrow[d, "\Pi_{01}"] & H^2(\mathbb{C}) \arrow[d, "\Pi_{00}"] \\ H^0(\mathcal{C}^{02})/\overline{\partial}H^0(\mathcal{A}^{01}) \arrow[r, "\widetilde{\Delta}^0"] & H^1(\mathcal{C}^{01}) \arrow[r, "\widetilde{\Delta}^1"] & H^2(\mathcal{O}) \end{tikzcd} \tag{6.25}$$

Moreover, $\Delta^0, \Delta^1, \widetilde{\Delta}^0, \widetilde{\Delta}^1$ are isomorphisms (cf. §4, in particular (4.11)). We decompose the inclusion i in (6.9) by

$$\mathbb{Z} \xrightarrow{h} \mathbb{C} \xrightarrow{\Pi_{00}} \mathcal{O}, \tag{6.26}$$

so that $i = \Pi_{00} \circ h$. For any $\beta \in H^2(\mathbb{Z})$ we have then

$$i^2\beta = \Pi_{00} h\beta = \widetilde{\Delta}^1 \widetilde{\Delta}^0 \Pi_{02} (\Delta^0)^{-1} (\Delta^1)^{-1} h\beta,$$

by the commutativity of the diagram (6.25). If $\beta \in H^2_{(1,1)}(M, \mathbb{Z})$ we have

$$\Pi_{02}(\Delta^0)^{-1}(\Delta^1)^{-1}h\beta = 0,$$

so that $i^2\beta = 0$. This completes the proof of 6(B).

To study $H^1(M, \mathcal{O}^*)$ the next step is to consider the subgroup of all $E \in H^1(M, \mathcal{O}^*)$ such that $c(E) = 0$. By the exactness of the sequence (6.23) this is isomorphic to p. 54

$$H^1(M, \mathcal{O})/i^1H^1(M, \mathbb{Z}). \tag{6.27}$$

For a nonsingular projective variety M the group (6.27) is compact and is called the *Picard variety* of M, [9].

The following are some important examples of holomorphic line bundles.

6.1. **Example.** The determinant bundle $\bigwedge^q(E)$ of a holomorphic vector bundle E of fibre dimension q. If g_{UV} are the transition functions of E, so that g_{UV} are nonsingular $(q\times q)$-matrices with elements which are holomorphic functions in $U \cap V$, the bundle $\bigwedge^q(E)$ is defined by the transition functions $\det g_{UV}$. If T^* is the cotangent bundle of M and $\dim M = m$, then $\bigwedge^m(T^*)$ is called the *canonical bundle* of M; it will be denoted as $K(M)$. If $z^1_U, \ldots, z^m_U$ and $z^1_V, \ldots, z^m_V$ are the local coordinates in U and V respectively, the transition functions of $K(M)$ are the jacobian determinants

$$k_{UV} = \frac{\partial\left(z^1_U, \ldots, z^m_U\right)}{\partial\left(z^1_V, \ldots, z^m_V\right)} \tag{6.28}$$

6.2. **Example.** Consider the line bundle in Example 1.2. We will call it the *universal line bundle* over $\mathbb{P}_m$. Here the base space $\mathbb{P}_m$ has the covering $\{U_i\}$, and the bundle has the transition functions $g_{ij} = {}_j\xi^i = \frac{z^i}{z^j}$, $0 \le i, j \le m$, $i \neq j$. The linear form $\sum a_i z^i$ in $\mathbb{C}_{m+1} - 0$, where the a's are constants, has in the local coordinates in $\psi^{-1}(U_i)$ the expression

$$\sum_j a_j z^j = z^i \left(a_{0_i}\xi_0 + \cdots + \underset{i\text{th}}{1} + \cdots + a_{m_i}\xi_m \right).$$

The expression in parentheses, which is essentially the linear form at the left-hand side in "nonhomogeneous" coordinates in U_i, defines a section in the line bundle whose transition functions are $g'_{ij} = \frac{z^j}{z^i} = \left({}_j\zeta^i\right)^{-1}$. Because of this origin the latter p. 55 bundle, to be denoted by H, is called the *hyperplane section bundle* of $\mathbb{P}_m$; it is the negative or dual of the universal line bundle.

Moreover, a holomorphically immersed submanifold $f : M \to \mathbb{P}_m$ has an induced bundle f^*H, called the *hyperplane section bundle* of M.

7. Hermitian geometry and kählerian geometry

Let M be a complex manifold of dimension m. M is called *hermitian* if an hermitian structure H is given in its tangent bundle $T(M)$. With the local coordinates $z^1, \ldots, z^m$ a natural frame field is given by

$$s_i = \frac{\partial}{\partial z^i}, \quad 1 \leq i, j, k, \ell \leq m, \tag{7.1}$$

and this frame is holomorphic. Let

$$h_{ik} = H\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^k}\right) = \overline{h}_{ki}. \tag{7.2}$$

Then the matrix

$$H = {}^t\overline{H} = (h_{ik}) \tag{7.3}$$

is positive definite hermitian.

There are special features arising from the fact that the bundle in question is the tangent bundle. First there is the Kähler form (cf. (2.16))

$$\widehat{H} = \frac{i}{2}\sum_{j,k} h_{jk} \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^k, \tag{7.4}$$

which is a real-valued form of type $(1,1)$. An hermitian manifold is called *kählerian* if its Kähler form is closed:

$$\mathrm{d}\widehat{H} = 0. \tag{7.5}$$

Secondly, let s be a local frame field, holomorphic or not. To s there is uniquely p. 56
determined a coframe field $\sigma = (\sigma^1, \ldots, \sigma^m)$ such that at every point $x \in M$ the sections $s_1(x), \ldots, s_m(x)$ of s and the sections $\sigma^1(x), \ldots, \sigma^m(x)$ of σ in the cotangent bundle are dual bases. The sections σ^i, being in the cotangent bundle, are complex-valued 1-forms, and they are everywhere linearly independent. Let s' be a new frame field, related to s by (5.20), and let σ' be its dual coframe field. Write

$$\begin{aligned} {}^t s &= (s_1, \ldots, s_m), \quad s' = (s'_1, \ldots, s'_m), \\ \sigma &= (\sigma'^1, \ldots, \sigma'^m). \end{aligned} \tag{7.6}$$

If T_x and T^*_x are respectively the tangent and cotangent spaces at x, we denote their pairing by

$$\langle \xi, \omega \rangle, \quad \xi \in T_x,\ \omega \in T^*_x. \tag{7.7}$$

Then we have

$$\left\langle s_i, \sigma^k \right\rangle = \left\langle s'_i, \sigma'^k \right\rangle = \delta^k_i. \tag{7.8}$$

Equation (5.20) can be written

$$s'_i = \sum_j g^j_i s_j, \quad g = (g^j_i). \tag{7.9}$$

In view of (7.8) we have

$$\sigma^i = \sum_j g^i_j \sigma'^j \tag{7.10}$$

or, in matrix notation,

$$\sigma = \sigma' g. \tag{7.10a}$$

By taking the exterior derivative of (7.10a) and using (5.22), we get p. 57

$$(\mathrm{d}\sigma' - \sigma' \wedge \omega')g = \mathrm{d}\sigma - \sigma \wedge \omega. \tag{7.11}$$

We will call the $(1 \times q)$-matrix

$$\tau = \mathrm{d}\sigma - \sigma \wedge \omega \tag{7.12}$$

the *torsion matrix*. It is a matrix of complex-valued two-forms and follows the transformation law (7.11) under a change of the frame field, holomorphic or not.

Proposition 7(A). Let M be an hermitian manifold. A connection in its tangent bundle is of type $(1, 0)$ if and only if its torsion matrix is of type $(2, 0)$.

To prove this let σ be the dual coframe field of a holomorphic frame field s. The connection matrix ω relative to s can be written in a unique way as

$$\omega = \omega_1 + \omega_2,$$

where ω_1 and ω_2 are matrices of 1-forms of types $(1, 0)$ and $(0, 1)$ respectively. σ is a matrix of forms of type $(1, 0)$ with holomorphic coefficients. The torsion matrix τ in (7.12) is of type $(2, 0)$ if and only if

$$\sigma \wedge \omega_2 = 0.$$

Let

$$\sigma = (\sigma^1, \ldots, \sigma^m), \ \omega_2 = (\theta^k_i), \quad 1 \le i, j, k \le m.$$

Then the above relation can be written explicitly as

$$\sum_i \sigma^i \wedge \theta^k_i = 0.$$

Since σ^i are linearly independent, we have

$$\theta^k_i = \sum_j a^k_{ij} \sigma^j,$$

where a^k_{ij} are functions. But σ^j is of type $(1, 0)$, while θ^k_i is of type $(0, 1)$ by our p. 58
hypothesis. It follows that the above relation is equivalent to

$$\theta^k_i = 0 \quad \text{or} \quad \omega_2 = 0.$$

This proves 7(A).

The criterion expressed by 7(A) has the advantage that, unlike the notion of a connection of type $(1, 0)$ which is defined in terms of holomorphic frame fields, it has a meaning for C^∞ frame fields. In the study of hermitian manifolds it is desirable to use C^∞ frame fields, for example, the unitary frame fields. It follows from §6 and 7(A) that an hermitian manifold has a uniquely determined admissible connection in its tangent bundle whose torsion matrix is of type $(2,0)$. When we speak of the connection in an hermitian manifold, this will be the connection meant. It is to be noticed that the curvature matrix of this connection is of type $(1,1)$ (relative to C^∞ frame fields).

Proposition 7(B). An hermitian manifold is kählerian if and only if the torsion matrix of its connection is zero.

Since both properties are independent of the choice of a frame field, it suffices to verify this theorem by using the natural frame field (7.1). Its dual coframe field is

$$\sigma = (dz^1, \dots, dz^m),$$

so that $\mathrm{d}\sigma = 0$. By (6.2) the vanishing of the torsion matrix can be written

$$\sigma \wedge \partial H = 0,$$

or, in expanded form,

$$\sum_{i,j} \frac{\partial h_{ik}}{\partial z^j} \mathrm{d}z^j \wedge \mathrm{d}z^i = 0, \qquad 1 \le i, j, k \le m.$$

The latter is equivalent to the conditions p. 59

$$\frac{\partial h_{ik}}{\partial z^j} - \frac{\partial h_{jk}}{\partial z^i} = 0. \tag{7.13}$$

One sees directly that (7.5) and (7.13) are equivalent.

Proposition 7(C). An hermitian manifold is kählerian if and only if there exists locally a real-valued C^∞ function u, such that its Kähler form can be written

$$\widehat{H} = i\partial\overline{\partial}u. \tag{7.14}$$

It suffices to prove that a kählerian manifold has the property stated in the theorem, for the form (7.14) is clearly closed. Suppose therefore that $\widehat{H}$ is closed. There exists locally a real-valued 1-form ω such that

$$\widehat{H} = \mathrm{d}\omega.$$

We can write

$$\omega = \alpha + \overline{\alpha},$$

where

$$alpha = \Pi_{1,0}\omega, \qquad \overline{\alpha} = \Pi_{0,1}\omega.$$

Then

$$\mathrm{d}\omega = \partial\alpha + (\overline{\partial}\alpha + \partial\overline{\alpha}) + \overline{\partial}\overline{\alpha},$$

where the terms are of types $(2,0)$, $(1,1)$, $(0,2)$ respectively. Since $\mathrm{d}\omega$ is of type $(1,1)$, we have

$$\overline{\partial}\overline{\alpha} = 0.$$

It follows from the Dolbeault–Grothendieck lemma that there exists a complex-valued C^∞-function F such that

$$\bar{\alpha} = \bar{\partial} F.$$

Then p. 60

$$\widehat{H} = \mathrm{d}\omega = \partial\bar{\partial}(F - \bar{F}).$$

The theorem follows by setting $u = -i(F - \bar{F})$.

The most important local properties of an hermitian manifold arise from its curvature matrix. The latter is defined in terms of a frame field. To have the situation under control we list together the formulas giving the effect from a change of the frame field on the various matrices we have introduced (formulas (5.20), (5.23), (5.53), (7.10a)):

$$\begin{aligned} s' &= gs, \\ \Omega' g &= g\Omega, \\ H' &= gH{}^t\bar{g}, \\ \sigma &= \sigma' g. \end{aligned} \tag{7.15}$$

From the second and third formulas of (7.15) we get

$$\Omega' H' = g\Omega H{}^t\bar{g}. \tag{7.16}$$

We note that ΩH is skew-hermitian (cf. (5.56)).

Since ΩH is of type $(1,1)$, we set

$$\begin{aligned} \Omega H &= (\Omega_{ik}), \\ \Omega_{ik} &= \sum_{j,\ell} R_{ikj\ell}\sigma^j \wedge \bar{\sigma}^\ell. \end{aligned} \tag{7.17}$$

The skew-hermitian property of ΩH is then expressed by

$$R_{ikj\ell} = \bar{R}_{ki\ell j}. \tag{7.18}$$

Throughout this part of our discussion we suppose as usual that our small Latin indices have the range from 1 to m:

$$1 \le i, j, k, \ell, p, q, u, v \le m. \tag{7.19}$$

The fourth equation of (7.15) and the equation (7.16) can be written out in detail as follows: p. 61

$$\sigma^i = \sum_j g^i_j \sigma'^j,$$

$$\sum_{j,\ell} R'_{ikj\ell}\sigma'^j \wedge \bar{\sigma}'^\ell = \sum_{p,q,u,v} g^p_i \bar{g}^q_k R_{pquv}\sigma^u \wedge \bar{\sigma}^v,$$

where the left-hand sides of the second equation are the entries in the matrix $\Omega' H'$. It follows that

$$R'_{ikj\ell} = \sum_{p,q,u,v} g^p_i \bar{g}^q_k g^u_j \bar{g}^v_\ell R_{pquv}. \tag{7.20}$$

Let

(7.21) $$\xi = \sum_i \xi^i s_i = \sum_j \xi'^j s'_j$$

be a vector at $x \in M$. The ξ^i and ξ'^j in (7.21) are the components of the vector relative to the frames s and s' respectively. Between them we have the relation

(7.22) $$\xi^i = \sum_j g^i_j \xi'^j.$$

From (7.20) and (7.22) we get

$$\sum_{i,\dots,\ell} R'_{ikj\ell} \xi'^i \overline{\xi}'^k \xi'^j \overline{\xi}'^\ell = \sum_{i,\dots,\ell} R_{ikj\ell} \xi^i \overline{\xi}^k \xi^j \overline{\xi}^\ell,$$

so that the common expression is independent of the choice of the frame field. If $\xi \neq 0$, we define

(7.23) $$R(x,\xi) = 2\frac{\sum_{i,\dots,\ell} R_{ikj\ell} \xi^i \overline{\xi}^k \xi^j \overline{\xi}^\ell}{\left(\sum_{i,k} h_{ik} \xi^i \overline{\xi}^k\right)^2}$$

to be the *holomorphic sectional curvature* at (x, ξ).

From the second equation of (7.15) we get

(7.24) $$\operatorname{Tr} \Omega' = \operatorname{Tr} \Omega = \phi \quad \text{(say)}.$$

ϕ is a form of type $(1,1)$ and is called the *Ricci form* of the hermitian metric. The metric is called *hermitian-einsteinian* if the Ricci form is a multiple of the Kähler form. p. 62

Let h^{ik} be the elements of the matrix H^{-1}. By the symmetry relations (7.18) we see that

(7.25) $$R = \sum_{i,\dots,\ell} R_{ikj\ell} h^{ik} h^{j\ell}$$

is real; it is also independent of the choice of the frame field. This quantity R is called the *scalar curvature*.

Compact kählerian manifolds have strong topological restrictions. Perhaps the simplest among them is the following:

Proposition 7(D). The second Betti number of a compact kählerian manifold is positive.

Corollary. The Hopf and Calabi–Eckmann manifolds $S^{2p+1} \times S^{2q+1}$, $p \geq 0$, $q \geq 1$, cannot be given a kählerian structure. (Cf. §1.)

Since $\widehat{H}$ is closed, it determines by de Rham's theorem an element $u \in H^2(M, \mathbb{R})$. To prove 7(D) we make use of the fact that the $2m$-form

$$\widehat{H}^m = \widehat{H} \wedge \cdots \wedge \widehat{H}, \quad m \text{ times}$$

determines the element $u^m = u \cup \cdots \cup u$ (cup product m times) of $H^{2m}(M, \mathbb{R})$. Using the local expression (7.4), we find

$$\widehat{H}^m = \left(\frac{i}{2}\right)^m m!(\det H) \bigwedge_j \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^j. \tag{7.26}$$

Since the matrix H is positive definite, $\det H > 0$. It follows that

$$\int_M \widehat{H}^m > 0,$$

and $u^m \neq 0$. Therefore $u \neq 0$. p. 63

Let M, N be complex manifolds, of dimensions m, n respectively. A continuous mapping $f : M \to N$ is called *holomorphic*, if locally it is defined by expressing the coordinates of the image point as holomorphic functions of those of the original point. f is called an *immersion*, if $m \leq n$ and if the jacobian matrix is of rank m everywhere. An immersion f is called an *embedding*, if it is one-one, i.e., if $f(x) = f(y)$, $x, y \in M$, implies $x = y$. The following is immediate:

Theorem 7(E). Let N be a kählerian manifold and let $f : M \to N$ be a holomorphic immersion. Then M has a kählerian structure.

Consider the complex projection space $\mathbb{P}_n$ of dimension n. It is known that (cf. §8)

$$\begin{aligned} H^{2i}(\mathbb{P}_n, \mathbb{Z}) &\cong \mathbb{Z}, \quad 0 \leq i \leq n, \\ H^k(\mathbb{P}_n, \mathbb{Z}) &= 0, \quad k \text{ odd}. \end{aligned} \tag{7.27}$$

Moreover we will show in §8 that $\mathbb{P}_n$ is kählerian. In this case, however, there is an additional important fact: The cohomology group $H^2(\mathbb{P}_n, \mathbb{R})$ is a real vector space of real dimension 1 and is isomorphic to $jH^2(\mathbb{P}_n, \mathbb{Z}) \otimes \mathbb{R}$, where j is induced by the coefficient homomorphism $j : \mathbb{Z} \to \mathbb{R}$. In other words, if ζ denotes a generator of $H^2(\mathbb{P}_n, \mathbb{Z})$, $j\zeta$ generates $H^2(\mathbb{P}_n, \mathbb{R})$ over $\mathbb{R}$. By the multiplication of a constant factor when necessary, we can define on $\mathbb{P}_n$ a kählerian metric such that the cohomology class $u \in H^2(\mathbb{P}_n, \mathbb{R})$ determined by the Kähler form belongs to $jH^2(\mathbb{P}_n, \mathbb{Z})$. A kählerian manifold with this property is said to be *of restricted type*.

Under the conditions of Theorem 7(E) we have the commutative diagram p. 64

$$\begin{array}{ccc} H^2(N, \mathbb{Z}) & \xrightarrow{j} & H^2(N, \mathbb{R}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^2(M, \mathbb{Z}) & \xrightarrow{j} & H^2(M, \mathbb{R}) \end{array}$$

where f^* is induced by the mapping f and j is induced by the coefficient homomorphism. Theorem 7(E) has the following complement:

Theorem 7(E'). Let N be a kählerian manifold of restricted type and let $f : M \to N$ be a holomorphic immersion. Then M is a kählerian manifold of restricted type.

A theorem of Chow says that a compact complex manifold holomorphically embedded in $\mathbb{P}_n$ is an algebraic variety, i.e., its locus is defined by a finite number of polynomial equations. The embedding theorem of Kodaira says that a compact kählerian manifold of restricted type can be holomorphically embedded in a projective space, [8].

As an example we will study the conditions that the complex torus $\Theta = \mathbb{C}_m/\Gamma$ (Ex. 1.4) can be given a kählerian structure of restricted type. Suppose that such a kählerian structure exists on Θ. The latter being a compact connected Lie group, we can integrate the kählerian metric over Θ. The resulting metric will define a kählerian structure of restricted type which is invariant under the action of Θ.

Let $z^1, \dots, z^m$ be the coordinates in $\mathbb{C}_m$, and let Γ be generated by the vectors

$$\pi_\lambda = \left(\pi_\lambda^1, \dots, \pi_\lambda^m\right), \quad 1 \le \lambda, \mu \le 2m, \tag{7.28}$$

which are linearly independent over $\mathbb{R}$. Let the kählerian structure of restricted type be given by

$$\mathrm{d}s^2 = \sum_{i,k} h_{ik} \mathrm{d}z^i \mathrm{d}\bar{z}^k, \quad 1 \le i, j, k \le m, \tag{7.29}$$

where h_{ik} are C^∞ functions in Θ. If the structure is invariant, as we are allowed to assume, h_{ik} are constants. A homology basis for the two-dimensional cycles of Θ is formed by the two-dimensional tori $\tau_{\lambda\mu}$, $\lambda < \mu$; $\tau_{\lambda\mu}$ is the quotient of the space spanned by π_λ, π_μ divided by the discrete group generated by π_λ, π_μ. It follows that the metric (7.29) is of restricted type, if and only if p. 65

$$g_{\lambda\mu} = -g_{\mu\lambda} = i \sum_{j,k} h_{jk} \left(\pi_\lambda^j \bar{\pi}_\mu^k - \pi_\mu^j \bar{\pi}_\lambda^k\right) \tag{7.30}$$

are integers. We introduce the matrices

$$G = \left(g_{\lambda\mu}\right), \quad H = \left(h_{ik}\right), \quad \Pi = \left(\pi_\lambda^k\right), \tag{7.31}$$

so that their orders are $(2m \times 2m)$, $(m \times m)$, and $(2m \times m)$ respectively. Then (7.30) can be written

$$\begin{aligned} G &= \sqrt{-1}\left(\Pi H^t \bar{\Pi} - \bar{\Pi} H^t \Pi\right) \\ &= (\Pi, \bar{\Pi}) \begin{pmatrix} \sqrt{-1} H & 0 \\ 0 & -\sqrt{-1}\,\bar{H} \end{pmatrix} \begin{pmatrix} {}^t\bar{\Pi} \\ {}^t\Pi \end{pmatrix}. \end{aligned} \tag{7.30a}$$

Taking the inverse matrix of this equation, we get

$$\begin{pmatrix} {}^t\bar{\Pi} \\ {}^t\Pi \end{pmatrix} G^{-1} (\Pi, \bar{\Pi}) = \begin{pmatrix} -\sqrt{-1} H^{-1} & 0 \\ 0 & \sqrt{-1}\,\bar{H}^{-1} \end{pmatrix}.$$

Therefore we have the:

Theorem 7(F). A necessary and sufficient condition for the torus $\Theta = \mathbb{C}_m/\Gamma$ to have a kählerian metric of restricted type is that there exists a skew-symmetric matrix G with integral elements such that p. 66

(7.32)
$$i\,{}^t\overline{\Pi}G^{-1}\Pi > 0,$$
$${}^t\Pi G^{-1}\Pi = 0.$$

The first condition in (7.32) means that the hermitian matrix at the left-hand side is positive definite.

The conditions (7.32) were first given by Riemann.

We wish to simplify the Riemann conditions (7.32) by proper choices of the basis vectors of $\mathbb{C}_m$ and of Γ. Let

$$(z^1, \dots, z^m) \to (\widetilde{z}^1, \dots, \widetilde{z}^m) = (z^1, \dots, z^m)T$$

be a change of coordinates in $\mathbb{C}_m$, where T is a nonsingular $(m \times m)$-matrix with complex elements. Meanwhile, under a change of basis of Γ the matrix Π is transformed according to

$$\Pi \to \widetilde{\Pi} = U\Pi,$$

where U is a unimodular integral matrix. The combined effect of these changes on the matrices is given by

(7.33)
$$\Pi \to \widetilde{\Pi} = U\Pi T,$$
$$H \to \widetilde{H} = T^{-1}H\,{}^t\overline{T}^{-1},$$
$$G \to \widetilde{G} = UG\,{}^tU.$$

It is known in the theory of matrices that U can be so chosen that

(7.34)
$$\widetilde{G} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where p. 67

(7.35)
$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix}, \quad d_i \in \mathbb{Z}.$$

We can then choose T, so that

(7.36)
$$\widetilde{\Pi} = \begin{pmatrix} I \\ \Sigma \end{pmatrix}.$$

With $\widetilde{G}, \widetilde{\Pi}$ in place of G, Π, the conditions (7.32) become

(7.37)
$${}^tZ = Z, \quad i(\overline{Z} - Z) > 0,$$

where

(7.38)
$$Z = \Sigma D.$$

The domain defined by (7.37) is of dimension $\frac{m(m+1)}{2}$; it is called the *Siegel upper half plane* and is known to be biholomorphically equivalent to one of the bounded symmetric domains of Élie Cartan. For $m = 1$ it is the Poincaré half-plane.

An example of a torus not satisfying the Riemann conditions is given in the case $m = 2$ by

$$\Sigma = \begin{pmatrix} \sqrt{-2} & \sqrt{-5} \\ \sqrt{-3} & \sqrt{-7} \end{pmatrix}. \tag{7.39}$$

Then

$$\Sigma D = \begin{pmatrix} \sqrt{-2}d_1 & \sqrt{-5}d_2 \\ \sqrt{-3}d_1 & \sqrt{-7}d_2 \end{pmatrix}, \quad d_i \in \mathbb{Z}.$$

For this matrix to be symmetric we must have $d_1 = d_2 = 0$. Then the second condition in (7.37) will not be fulfilled. Thus the corresponding torus will not have a kählerian structure of restricted type.

In the general case a meromorphic function on the complex torus Θ is p. 68
identical to a $2m$-ply periodic meromorphic function in $\mathbb{C}_m$ with the period vectors (7.28). All the meromorphic functions on Θ form a field. It follows from Kodaira's embedding theorem that a complex torus satisfying the conditions (7.32) can be holomorphically embedded in a projective space and is thus by definition an *abelian variety*. At every point x of an abelian variety Θ of dimension m there are m meromorphic functions on Θ, which are functionally independent at x. On the other hand, there exist complex tori on which every meromorphic function is a constant.

8. The Grassmann manifold

Let

$$\mathbb{C}_{N+1} = \mathbb{C} \times \cdots \times \mathbb{C}, \qquad N+1 \text{ factors} \tag{8.1}$$

be the complex number space of $N+1$ dimensions. Let $\mathrm{GL}(N+1, \mathbb{C})$ be the general linear group in $N+1$ complex variables, which we identify with the group of all $(N+1) \times (N+1)$ nonsingular matrices with complex elements. Suppose $\mathrm{GL}(N+1, \mathbb{C})$ acts on $\mathbb{C}_{N+1}$ to the right, as described by

$$(z^0, \ldots, z^N) \to (z^0, \ldots, z^N)g, \quad g \in \mathrm{GL}(N+1, \mathbb{C}). \tag{8.2}$$

Among the subgroups of $\mathrm{GL}(N+1, \mathbb{C})$ are:

(1) the unitary group $\mathrm{U}(N+1)$, which consists of all matrices g satisfying

$$^t g\bar{g} = I, \tag{8.3}$$

where I is the identity matrix;

(2) the group $\mathrm{GL}(k+1, N-k, \mathbb{C})$, consisting of all nonsingular matrices of the form p. 69

$$\begin{array}{cc} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} & \begin{array}{l} \} \, k+1 \\ \} \, N-k \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{k+1} & \underbrace{\quad}_{N-k} \end{array} & \end{array} \tag{8.4}$$

where the elements at the upper-right corner are zero. The group $\mathrm{GL}(k+1, N-k, \mathbb{C})$ is the subgroup of all elements of $\mathrm{GL}(N+1, \mathbb{C})$ leaving fixed the $(k+1)$-dimensional subspace of C_{N+1} spanned by the first $k+1$ coordinate vectors.

The space of all $(k+1)$-dimensional linear subspaces of $\mathbb{C}_{N+1}$, $k \geq 0$, is called a *Grassmann manifold*, to be denoted by $\mathrm{Gr}(N, k)$. Using the projection

$$\psi : \mathbb{C}_{N+1} - 0 \longrightarrow \mathbb{P}_N \tag{8.5}$$

in Example 1.2, the notation suggests that it is also the space of all k-dimensional linear (projective) subspaces in $\mathbb{P}_N$.

From the above discussion $\mathrm{Gr}(N, k)$ can be represented as a right coset space in two different ways:

$$\mathrm{Gr}(N, k) = \frac{\mathrm{GL}(N+1, \mathbb{C})}{\mathrm{GL}(k+1, N-k, \mathbb{C})} = \frac{\mathrm{U}(N+1)}{\mathrm{U}(k+1) \times \mathrm{U}(N-k)}. \tag{8.6}$$

The first representation shows that it is a complex manifold of dimension $(k+1)(N-k)$. The second representation shows that it is compact.

An element of $\mathrm{Gr}(N, k)$ can be given by a nonzero decomposable $(k+1)$-vector

$$\Lambda = X_0 \wedge X_1 \wedge \cdots \wedge X_k \neq 0 \tag{8.7}$$

defined up to a constant factor. If $e_0^0, \ldots, e_N^0$ denote a fixed frame in $\mathbb{C}_{N+1}$, we can write

$$\Lambda = \sum_{\alpha} P_{\alpha_0,\ldots,\alpha_k} e_{\alpha_0}^0 \wedge \cdots \wedge e_{\alpha_k}^0, \quad 0 \leq \alpha_0, \ldots, \alpha_k \leq N, \tag{8.8}$$

where the P's are skew-symmetric in their indices. The $P_{\alpha_0 \cdots \alpha_k}$ are called the *Cayley–Plücker–Grassmann* coordinates in $\mathrm{Gr}(N, k)$. By considering $P_{\alpha_0 \cdots \alpha_k}$ as the homogeneous coordinates of a projective space of dimension $\binom{N+1}{k+1} - 1$, we get an embedding of $\mathrm{Gr}(N, k)$ in the latter. p. 70

We propose to study the topological properties of $\mathrm{Gr}(N, k)$. Our main step is to obtain a cell decomposition of $\mathrm{Gr}(N, k)$ by means of the *Schubert varieties*. This was first accomplished by C. Ehresmann in 1934. Let

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_k \leq N - k \tag{8.9}$$

be a sequence of integers, and let

$$L_{a_0} \subset L_{a_1+1} \subset \cdots \subset L_{a_k+k} \subset \mathbb{P}_N \tag{8.10}$$

be a nested sequence of linear spaces whose dimensions are given by the subscripts. (We will deal with linear spaces of $\mathbb{P}_N$; their images under ψ^{-1} will have one dimension higher.) A *Schubert variety* $(a_0 a_1 \cdots a_k)$ is the set of all k-dimensional linear spaces $X \in \mathrm{Gr}(N, k)$ such that

$$\dim\left(X \cap L_{a_j+j}\right) \geq j, \quad 0 \leq j \leq k. \tag{8.11}$$

By definition, $(a_0a_1\cdots a_k)$ is a closed subset of $\mathrm{Gr}(N,k)$ and is determined by the a's up to a projective collineation. Its (complex) dimension is

$$\dim(a_0a_1\cdots a_k) = a_0 + a_1 + \cdots + a_k. \tag{8.12}$$

Examples. (1) $(N-k\cdots N-k) = \mathrm{Gr}(N,k)$; p. 71

(2) $(0\cdots 0) = L_k$;

(3) $(0\ldots 01\ldots 1)$ (r ones) is the set of all X satisfying the condition

$$L_{k-r} \subset X \subset L_{k+1}, \tag{8.13}$$

where L_{k-r} and L_{k+1} are fixed linear spaces of dimensions $k-r$ and $k+1$ respectively.

We take a fixed sequence of linear spaces in $\mathbb{P}_N$:

$$L_0 \subset L_1 \subset \cdots \subset L_{N-1} \subset \mathbb{P}_N \tag{8.14}$$

and suppose the Schubert varieties are constructed from the linear spaces of this sequence. Put

$$(a_0\cdots a_k)^* = (a_0\cdots a_k) - \sum_{a_{j-1}<a_j} (a_0\cdots a_{j-1}a_j-1\cdots a_k), \qquad (a_{-1}=0). \tag{8.15}$$

Then any $X \in \mathrm{Gr}(N,k)$ belongs to a unique $(a_0\cdots a_k)^*$. That is, the sets $(a_0\cdots a_k)^*$ are mutually disjoint and their union is $\mathrm{Gr}(N,k)$.

Proposition 8(A). $(a_0\cdots a_k)^*$ is an open cell of real dimension $2(a_0+\cdots+a_k)$.

Example. For $k=0$ we have

$$\begin{aligned} \mathbb{P}_N &= (N)^* + (N-1)^* + \cdots + (1)^* + (0)^* \\ &= (\mathbb{P}_N - L_{N-1}) + (L_{N-1} - L_{N-2}) + \cdots + (L_1 - L_0) + L_0. \end{aligned} \tag{8.16}$$

That is, $\mathbb{P}_N$ is a union of cells of dimensions $0, 2, 4, \ldots, 2N$ respectively, which are mutually disjoint.

We prove 8(A) by induction on k. The example shows that it is true for $k=0$. p. 72
For definiteness we suppose $a_0 > 0$; the treatment of the case $a_0 = 0$ requires only slight modifications. We take a hyperplane π in $\mathbb{P}_N$ such that

$$\begin{aligned} \pi \cap L_{a_0} &= L_{a_0-1}, \\ \pi \cap L_q &= L'_{q-1}, \quad a_0 < q, \end{aligned}$$

where L'_{q-1} is of dimension $q-1$. Consider the set

$$\Sigma = (a_0a_1\cdots a_k) - (a_0-1a_1\cdots a_k).$$

If $X \in \Sigma$, it meets L_{a_0}, but not L_{a_0-1}. Hence it meets $L_{a_0} - L_{a_0-1}$ in exactly one point y (say). Moreover the intersection $\xi = X \cap \pi$ is of dimension $k-1$, satisfying $\xi \cap L_{a_0-1} = \varnothing$. We therefore have the continuous mapping

$$\phi : \Sigma \longrightarrow \left(L_{a_0} - L_{a_0-1}\right) \times A \tag{8.17}$$

defined by

$$\phi(X) = (y, \xi),$$

where A is the subset of the Grassmann manifold $\mathrm{Gr}(N-1, k-1)$ of all $(k-1)$-dimensional linear spaces in π such that $\xi \cap L_{a_0-1} = \varnothing$.

To describe the image $\phi(\Sigma)$ we consider in π the sequence of linear spaces

$$L_{a_0-1} \subset L'_{a_0} \subset \cdots \subset L'_{N-2} \subset \pi. \tag{8.18}$$

The Schubert varieties of $\mathrm{Gr}(N-1, k-1)$ to be considered will be defined relative to the sequence (8.18) and will be denoted by the same symbols with dashes. We have, for $j \geq 1$, p. 73

$$\dim\left(\xi \cap L'_{a_j+j-1}\right) = \dim\left(X \cap \pi \cap L_{a_j+j}\right) \geq j-1,$$

so that $\xi \in (a_1 \cdots a_k)'$. Conversely, if

$$(y, \xi) \in \left(L_{a_0} - L_{a_0-1}\right) \times (a_1 \cdots a_k)',$$

they span a k-dimensional space X whose intersection

$$X \cap L_{a_j+j} \quad (j \geq 1)$$

contains y and $\xi \cap L'_{a_j+j-1}$ and is hence of dimension $\geq j$. If moreover, $\xi \in A$, then $X \in \Sigma$. Thus ϕ is a homeomorphism of Σ onto the set

$$\left(L_{a_0} - L_{a_0-1}\right) \times \{((a_1 \cdots a_k)' \cap A\}.$$

Analogous to (8.15) we set

$$(a_1 \cdots a_k)'^* = (a_1 \cdots a_k)' - \sum_{a_{j-1} < a_j} (a_1 \cdots a_{j-1} a_j - 1 \cdots a_k)'.$$

Then $\xi \in (a_1 \cdots a_k)'^*$ implies that $\xi \cap L'_{a_1-1} = \varnothing$ so that $\xi \in A$. Our induction hypothesis says that $(a_1 \cdots a_k)'^*$ is an open cell.

The homeomorphism ϕ depends only on a_0 and on the choice of π; it is independent of the integers $a_1, \cdots, a_k$, provided that the conditions (8.9) are fulfilled. It follows that, for $j \geq 1$, $a_{j-1} < a_j$, ϕ restricts to a homeomorphism

$$\begin{array}{c} \phi : (a_0 a_1 \cdots a_{j-1} a_j - 1 \cdots a_k) - (a_0 - 1 a_1 \cdots a_{j-1} a_j - 1 \cdots a_k) \\ \Big\downarrow \\ \left(L_{a_0} - L_{a_0-1}\right) \times \left\{(a_1 \cdots a_{j-1} a_j - 1 \cdots a_k)' \cap A\right\}. \end{array}$$

From this we see easily that ϕ establishes a homeomorphism between $(a_0 \cdots a_k)^*$ and $\left(L_{a_0} - L_{a_0-1}\right) \times (a_1 \cdots a_k)'^*$. This proves 8(A).

The Schubert varieties relative to the sequence (8.14) give a cell decomposition of $\mathrm{Gr}(N, k)$ whose cells are all of even dimensions. From known theorems in algebraic topology (cf. [7]) we are thus able to draw the following conclusions on the topological properties of $\mathrm{Gr}(N, k)$: p. 74

Proposition 8(B). The Schubert varieties are cycles. $\mathrm{Gr}(N, k)$ is simply connected. It has no torsion coefficients and its homology groups of odd dimensions are zero. A homology basis of $\mathrm{Gr}(N, k)$ of dimension $2r$ is formed by the Schubert varieties $(a_0 a_1 \cdots a_k)$, where $a_0, a_1, \ldots, a_k$ run over all sets of integers satisfying $0 \leq a_0 \leq a_1 \leq \cdots \leq a_k \leq N-k$ and $a_0 + a_1 + \cdots + a_k = r$.

Example. $\mathrm{Gr}(3,1)$ is the space of all lines in $\mathbb{P}_3$. Its (complex) dimension is 4. Its Schubert cycles of different dimensions are respectively

$$(00), (01), (11), (02), (12), (22).$$

Hence its Betti numbers are

$$b^0 = b^8 = 1, \quad b^2 = b^6 = 1, \quad b^4 = 2,$$

where the superscripts indicate the (real) dimensions.

Of geometrical significance is the structure of the homology or cohomology rings of $\mathrm{Gr}(N,k)$, i.e., the intersection properties of the Schubert varieties. These are at the basis of enumerative geometry and have been completely determined (cf. [6], [4]). We will, however, not enter into this question.

The Grassmann manifold has been playing an important role in recent developments of mathematics, because it is a so-called *classifying space* for complex vector bundles. In fact, when $\mathrm{Gr}(N,k)$ is considered to be the manifold of all $(k+1)$-dimensional linear subspaces through the origin of $\mathbb{C}_{N+1}$, it is the base space of a complex vector bundle whose fibres are these linear subspaces themselves. More precisely, let $\Lambda \in \mathrm{Gr}(N,k)$ as defined by (8.7) and let $v \in \mathbb{C}_{N+1}$ be such that $v \wedge \Lambda = 0$, i.e., $v \in \Lambda$. Also let p. 75

(8.19) $$E_0 = \{(v, \Lambda) : v \wedge \Lambda = 0\}.$$

Then

(8.20) $$\psi_0 : E_0 \longrightarrow \mathrm{Gr}(N,k),$$

with the projection ψ_0 defined by

(8.21) $$\psi_0(v, \Lambda) = \Lambda,$$

is a complex vector bundle with the fibre dimension $k+1$.

The bundle (8.20) has an important property. To describe it we define a $(k+1)$-*frame* to be an ordered set of $k+1$ vectors $e_0, \ldots, e_k \in \mathbb{C}_{N+1}$ such that $e_0 \wedge \cdots \wedge e_k \neq 0$. The space of all $(k+1)$-frames in $\mathbb{C}_{N+1}$ is called a *Stiefel manifold*, to be denoted by $\mathrm{St}(N+1, k+1)$. It is the total space of a fibre bundle

(8.22) $$\lambda : \mathrm{St}(N+1, k+1) \longrightarrow E_0$$

over E_0, with the projection λ defined by

(8.23) $$\lambda(e_0, \ldots, e_k) = (e_0, e_0 \wedge \cdots \wedge e_k) \in E_0.$$

The total space $\mathrm{St}(N+1, k+1)$ has a string of vanishing homotopy groups expressed by

(8.24) $$\pi_i(\mathrm{St}(N+1, k+1)) = 0, \quad i \le 2N-2k$$

(cf. [11, p. 134]). Following Steenrod's terminology the bundle (8.20) is $(2N-2k+1)$-universal in the following sense: Let M be a compact manifold of real dimension $\le 2N-2k$. The equivalence classes of complex vector bundles of fibre dimension $k+1$ over M are in one-one correspondence with the homotopy classes of continuous mappings $f : M \longrightarrow \mathrm{Gr}(N,k)$, the correspondence being established by assigning to each mapping f the bundle f^*E_0 induced from E_0.

On account of this theorem the bundle (8.20) is called a *universal bundle* and its base space a *classifying space*. p. 76

The *universal rth Chern class* $\widetilde{c}_r$, $0 \leq r \leq k+1$, is the element of $H^{2r}(\mathrm{Gr}(N,k),\mathbb{Z})$ such that its value is 1 over the Schubert cycle $(0\cdots01\cdots1)$ (r ones) and is 0 over all other Schubert cycles. By the above theorem if $\psi : E \longrightarrow M$ is a complex vector bundle with fibre dimension $k+1$, it is induced from E_0 by a mapping $f : M \longrightarrow \mathrm{Gr}(N,k)$ (real dim $M \leq 2N-2k$), and f is defined up to a homotopy. It follows that $f^*\widetilde{c}_r \in H^{2r}(M,\mathbb{Z})$ is completely determined by the bundle E. We define

$$c_r(E) = f^*\widetilde{c}_r \in H^{2r}(M,\mathbb{Z}), \quad 0 \leq r \leq k+1; \tag{8.25}$$

$c_r(E)$ is called the *rth Chern class of E*. Clearly $c_0(E) = 1$.

In applications it will be essential to identify $c_r(E)$ with geometric or analytic invariants defined in other ways. We will sketch one such application without insisting on details. Let M be a compact almost complex manifold of real dimension $2n$. Its tangent bundle $T(M)$ is then a complex vector bundle over M with fibre dimension n. Then we have

$$c_n(T(M)) \cdot M = \chi(M), \tag{8.26}$$

where the left-hand side stands for the value of $c_n(T(M))$ on the fundamental cycle of M and the right-hand side $\chi(M)$ is the Euler–Poincaré characteristic of M.

To see this we consider the universal Chern class $\widetilde{c}_n \in H^{2n}(\mathrm{Gr}(N,n-1),\mathbb{Z})$, with N sufficiently large. By definition this is the class which has the value one over the Schubert cycle $(1\cdots1)$ (n ones) and the value zero over all other Schubert cycles. By Poincaré duality this can be realised by taking a fixed Schubert cycle $(N-n,\cdots,N-n)$ of complementary real dimension $2n(N-n)$ and taking its intersection with the Schubert cycles of real dimension $2n$. By p. 77 definition $(N-n\cdots N-n)$ consists of all the n-dimensional linear spaces through 0 in $\mathbb{C}_{N+1}$ which lie in a fixed hyperplane L of dimension N. Let v_0 be a vector through 0 in $\mathbb{C}_{N+1}$ orthogonal to L. By using the mapping $f : M \longrightarrow \mathrm{Gr}(N,n-1)$ and by taking the orthogonal projection of v_0 to $f(x)$, $x \in M$, we define a vector field over M, which will have singularities exactly at the points $x \in M$ such that $f(x) \in L$. One verifies that $c_n(T(M)) \cdot M$ is equal to the sum of the indices at the singularities of a vector field with a finite number of singularities. This proves (8.26).

By studying the homotopy groups of the unitary group, Bott proved the theorem: Let E be a complex vector bundle of fibre dimension n over the $2n$-sphere S^{2n}. Then $c_n(E) \cdot S^{2n}$ is divisible by $(n-1)!$.

If S^{2n} has an almost complex structure and E is the tangent bundle, then by (8.26) $c_n(T(S^{2n})) \cdot S^{2n}$ is equal to 2, the Euler–Poincaré characteristic of S^{2n}. It follows from Bott's theorem that S^{2n} has an almost complex structure only when $n \leq 3$. On the other hand, it can be proved by a different method that S^4 does not have an almost complex structure. Thus S^2 and S^6 are the only even-dimensional spheres which have almost complex structures.

We now study the geometry in $\mathrm{Gr}(N, k)$. For this purpose it is necessary to introduce an hermitian structure in the bundle (8.20). This is most easily achieved by introducing in $\mathbb{C}_{N+1}$ the hermitian scalar product

$$(Z, W) = \overline{(W, Z)} = z_0\overline{w}_0 + \cdots + z_N\overline{w}_N, \tag{8.27}$$

where

$$Z = (z_0, \ldots, z_N) \in \mathbb{C}_{N+1}, \qquad W = (w_0, \ldots, w_N) \in \mathbb{C}_{N+1}. \tag{8.28}$$

This induces an hermitian structure in E_0 in an obvious way.

The definition (8.27) can be extended to decomposable $(k + 1)$-vectors. In fact, let p. 78

$$\Lambda = X_0 \wedge \cdots \wedge X_k, \qquad M = Y_0 \wedge \cdots \wedge Y_k. \tag{8.29}$$

We define the hermitian scalar product

$$(\Lambda, M) = \det\Big(X_\alpha, Y_\beta\Big), \qquad 0 \leq \alpha, \beta \leq k. \tag{8.30}$$

The product (Λ, M) in (8.30) depends only on the $(k + 1)$-vectors and is independent of the ways that they are decomposed in (8.29). By the hermitian property of (Λ, M) and the fact that any $(k + 1)$-vector is a linear combination of decomposable $(k + 1)$-vectors, the definition of (Λ, M) is extended to arbitrary $(k + 1)$-vectors Λ, M. For simplicity of writing we will introduce the notations

$$|\Lambda, M| = |(\Lambda, M)|, \qquad |\Lambda| = +(\Lambda, \Lambda)^{\frac{1}{2}}. \tag{8.31}$$

$|\Lambda|$ is called the *norm* of Λ. Clearly the norm determines the scalar product. The quotient $\frac{|\Lambda, M|}{|\Lambda||M|}$ depends only on the elements of $\mathrm{Gr}(N, k)$ determined by the $(k + 1)$-vectors Λ, M, and we have the Schwarz inequality

$$|\Lambda, M| \leq |\Lambda||M|. \tag{8.32}$$

Utilising the scalar product (8.27) we will restrict ourselves to unitary frames. A *unitary* $(h+1)$*-frame* is an ordered set of $h+1$ vectors $Z_0, Z_1, \ldots, Z_h$ satisfying

$$(Z_i, Z_j) = \delta_{ij}, \qquad 0 \leq i, j \leq h. \tag{8.33}$$

If $h = N$, we will call it simply a *unitary frame*. We will identify $\mathrm{U}(N + 1)$ with the space of all unitary frames. Then we have the fibrings

$$\mathrm{U}(N + 1) \xrightarrow{\lambda} \mathrm{St}(N + 1, k + 1) \xrightarrow{\mu} \mathrm{Gr}(N, k), \tag{8.34}$$

where $\mathrm{St}(N+1, k+1)$ is the Stiefel manifold of all unitary $(k+1)$-frames in $\mathbb{C}_{N+1}$ and the projections λ, μ are defined by p. 79

$$\begin{aligned} \lambda(Z_0, Z_1, \ldots, Z_N) &= (Z_0, Z_1, \ldots, Z_k), \\ \mu(Z_0, Z_1, \ldots, Z_k) &= Z_0 \wedge Z_1 \wedge \cdots \wedge Z_k, \end{aligned} \tag{8.35}$$

the last $(k + 1)$-vector defining an element of $\mathrm{Gr}(N, k)$.

In $\mathrm{U}(N + 1)$ we put

$$\theta_{AB} = (\mathrm{d}Z_A, Z_B), \qquad 0 \leq A, B, C \leq N. \tag{8.36}$$

From the orthogonality relations (8.33) we get by differentiation

$$\theta_{AB} + \bar{\theta}_{BA} = 0. \tag{8.37}$$

Equation (8.36) can also be written

$$\mathrm{d}Z_A = \sum_B \theta_{AB} Z_B. \tag{8.38}$$

Taking its exterior derivative, we get

$$\mathrm{d}\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}. \tag{8.39}$$

These are called the *Maurer–Cartan equations* of the unitary group $\mathrm{U}(N+1)$.

Under the projection $\mu \circ \lambda$ in (8.34) the differential forms of $\mathrm{Gr}(N, k)$ are mapped into forms of $\mathrm{U}(N+1)$, and this mapping is an isomorphism, i.e., a form ω on $\mathrm{Gr}(N, k)$ is completely determined by its image $(\mu \circ \lambda)^*\omega$. We will utilise this fact by studying the forms on $\mathrm{U}(N+1)$ and consider a relation to be on $\mathrm{Gr}(N, k)$ when all the forms involved belong to the image of $(\mu \circ \lambda)^*$. Moreover for simplicity the mapping $(\mu \circ \lambda)^*$ will be omitted in the formulas.

With these conventions, let Λ be a decomposable $(k+1)$-vector and let

$$\Lambda_0 = \frac{\Lambda}{|\Lambda|}, \tag{8.40}$$

so that Λ_0 is a unit $(k+1)$-vector. We write p. 80

$$\Lambda_0 = Z_0 \wedge \cdots \wedge Z_k, \tag{8.41}$$

$Z_0, \dots, Z_k$ being a unitary $(k+1)$-frame. Then we get, by means of (8.38),

$$\begin{gathered}
(\mathrm{d}\Lambda_0, \Lambda_0) = \sum_\alpha \theta_{\alpha\alpha} = -\sum_\alpha \bar{\theta}_{\alpha\alpha}, \\
(\mathrm{d}\Lambda_0, \mathrm{d}\Lambda_0) = +\left(\sum_\alpha \theta_{\alpha\alpha}\right)\left(\sum_\alpha \bar{\theta}_{\alpha\alpha}\right) + \sum_{\alpha,r} \theta_{\alpha r}\bar{\theta}_{\alpha r}, \\
0 \le \alpha \le k, \quad k+1 \le r \le N,
\end{gathered} \tag{8.42}$$

where the multiplication of differential forms is in the sense of ordinary commutative multiplication. It follows that

$$(\mathrm{d}\Lambda_0, \mathrm{d}\Lambda_0) - (\mathrm{d}\Lambda_0, \Lambda_0)(\Lambda_0, \mathrm{d}\Lambda_0) = \sum_{\alpha,r} \theta_{\alpha r}\bar{\theta}_{\alpha r}.$$

By substituting the expression in (8.40), we get

$$\frac{1}{|\Lambda|^4}\{(\Lambda, \Lambda)(\mathrm{d}\Lambda, \mathrm{d}\Lambda) - (\mathrm{d}\Lambda, \Lambda)(\Lambda, \mathrm{d}\Lambda)\} = \sum_{\alpha,r} \theta_{\alpha r}\bar{\theta}_{\alpha r}. \tag{8.43}$$

This defines an hermitian structure in $\mathrm{Gr}(N, k)$. In fact, the left-hand side of (8.43) shows that it is hermitian and the right-hand side shows that the metric is positive definite.

The Kähler form of (8.43) is

$$\widehat{H}_k = \frac{i}{2}\sum_{\alpha,r}\theta_{\alpha r}\wedge\overline{\theta}_{\alpha r} = \frac{1}{2i}\sum_{\alpha,r}\theta_{\alpha r}\wedge\theta_{r\alpha} = \frac{1}{2i}\mathrm{d}\left(\sum_{\alpha}\theta_{\alpha\alpha}\right). \tag{8.44}$$

It is therefore closed, and the metric (8.43) is kählerian. By (8.40) and (8.42) we can further write

$$\sum_{\alpha}\theta_{\alpha\alpha} = (\partial - \overline{\partial})\log|\Lambda|, \tag{8.45}$$

so that

$$\widehat{H}_k = i\partial\overline{\partial}\log|\Lambda|. \tag{8.46}$$

We summarise the results in the: p. 81

Theorem 8(C). The Grassmann manifold $\mathrm{Gr}(N,k)$ has a kählerian structure invariant under the action of $\mathrm{U}(N+1)$. Its Kähler form is equal to π times the curvature form of the hyperplane section bundle over $\mathrm{Gr}(N,k)$ defined by the embedding by the Cayley–Plücker–Grassmann coordinates and the hermitian norm $|\Lambda|$.

The first statement has been proved. The second statement may need some explanation. All the $(k+1)$-vectors Λ of $\mathbb{C}_{N+1}$, decomposable or not, form a complex vector space $\mathbb{C}_\nu$ of dimension $\nu = \binom{N+1}{k+1}$. As in Example 1.2 and Example 6.2, $\mathbb{C}_\nu - \{0\} \longrightarrow \mathbb{P}_{\nu-1}$ defines the universal line bundle over $\mathbb{P}_{\nu-1}$ and an hermitian structure is introduced in this bundle by the norm $|\Lambda|$. The restriction of this bundle to $\mathrm{Gr}(N,k) \subset \mathbb{P}_{\nu-1}$ is the negative of the hyperplane section bundle meant in the theorem, and $|\Lambda|^{-1}$ defines an hermitian structure on it.

By (6.5) the curvature form of this bundle is $\frac{1}{2\pi i}\partial\overline{\partial}\log h$, where $h = |\Lambda|^{-2}$ is the square of the norm of a local holomorphic section. It is therefore equal to $\frac{i}{\pi}\partial\overline{\partial}\log|\Lambda| = +\frac{1}{\pi}\widehat{H}_k$, by (8.46).

This proves the second statement in 8(C).

Remark. Consider the universal bundle (8.20). Let V be a neighbourhood in $\mathrm{Gr}(N,k)$ and let Z_A, $0 \le A \le N$, be a frame field over V into $\mathrm{U}(N+1)$. Then $Z_0,\ldots,Z_k$ define a frame field of the bundle E_0 over V. The matrix $(\theta_{\alpha\beta})$, $0 \le \alpha,\beta \le k$, depends only on the $(k+1)$-frame field Z_α and follows the transformation law (5.22) under a change of the frame field. It therefore defines a connection in the bundle E_0. The curvature matrix of this connection is $\theta = (\theta_{\alpha\beta})$, where, by (8.39), p. 82

$$\begin{gathered}\theta_{\alpha\beta} = \mathrm{d}\theta_{\alpha\beta} - \sum_{\gamma}\theta_{\alpha\gamma}\wedge\theta_{\gamma\beta} = \sum\theta_{\alpha r}\wedge\theta_{r\beta} = -\sum\theta_{\alpha r}\wedge\overline{\theta}_{\beta r},\\ 0 \le \alpha,\beta,\gamma \le k,\ k+1 \le r \le N\end{gathered} \tag{8.47}$$

It follows that

$$\theta + {}^{t}\overline{\theta} = 0$$

and hence, as in (5.59), that the determinant $\det\left(I + \frac{i}{2\pi}\theta\right)$ is real. Using the notation of §5, we have therefore $(\mathrm{Im}\,P_r)(\theta) = 0$, $0 \le r \le k+1$.

By actual integration one can show that

$$\binom{k+1}{r}\int_{(0\cdots01\cdots1)}(\operatorname{Re}P_r)(\theta)=1, \tag{8.48}$$

where $(0\cdots01\cdots1)$ is the Schubert cycle with r ones and that the same integral over any other Schubert cycle is zero. This means that the element of $H^{2r}(\mathrm{Gr}(N,k),\mathbb{R})$ defined by $\binom{k+1}{r}(\operatorname{Re}P_r)(\theta)$ via the de Rham isomorphism is $j\widetilde{c}_r$, where $\widetilde{c}_r$ is the rth universal Chern class and

$$j:H^{2r}(\mathrm{Gr}(N,k),\mathbb{Z})\to H^{2r}(\mathrm{Gr}(N,k),\mathbb{R})$$

is induced by the coefficient homomorphism.

Let M be a compact manifold and $\psi:E\to M$ be a complex vector bundle of fibre dimension $k+1$ induced from E_0 by the mapping $f:M\to\mathrm{Gr}(N,k)$. Then the above relationship remains true for the induced connection. By Theorem 5(B), we conclude that

$$\binom{k+1}{r}P_r(\Omega),\qquad 0\le r\le k+1,$$

where Ω is the curvature matrix of any connection in E, corresponds to the Chern class $jc_r(E)$ by the de Rham isomorphism. This is a relationship between the curvature of a connection of a complex vector bundle and its characteristic classes and contains as a special case the Gauss–Bonnet formula in high dimensions. p. 83

9. Curves in a Grassmann manifold

As in §8 we will denote by $\mathrm{Gr}(N,k)$ the Grassmann manifold of all k-dimensional linear subspaces of the projective space $\mathbb{P}_N$ of dimension N. Let M be a one-dimensional complex manifold or Riemann surface. A *holomorphic curve* in $\mathrm{Gr}(N,k)$ is a holomorphic mapping $f:M\to\mathrm{Gr}(N,k)$. In particular, a holomorphic curve $f:M\to\mathrm{Gr}(1,0)=\mathbb{P}_1$ is a meromorphic function on M, given by the ratio of the homogeneous coordinates in $\mathbb{P}_1$.

Let Λ be a nonzero decomposable $(k+1)$-vector which defines an element of $\mathrm{Gr}(N,k)$, so that Λ is determined up to a factor. Let Σ_B be the subset of all $\Lambda\in\mathrm{Gr}(N,k)$ such that

$$(\Lambda,B)=0 \tag{9.1}$$

where B is a fixed $(k+1)$-vector, decomposable or not. Σ_B is then a submanifold of codimension one in $\mathrm{Gr}(N,k)$.

Appendix: Geometry of characteristic classes

1. Historical remarks and examples

2. Weil homomorphism

3. Secondary invariants

4. Vector fields and characteristic numbers

5. Holomorphic curves

References

[1] S. Chern. 'An elementary proof of the existence of isothermal parameters on a surface'. In: *Proceedings of the American Mathematical Society* 6 (1955), pp. 771–782. Zbl: 0066.15402.

[2] S. Chern and J. Simons. 'Characteristic forms and geometric invariants'. In: *Annals of Mathematics* 99 (1974), pp. 48–69. Zbl: 0283.53036.

[3] R. Godement. *Topologie algébrique et théorie des faisceaux*. Publications de l'Institut de Mathématique de l'Université de Strasbourg. Actualités Scientifiques et Industrielles 1252. Paris: Hermann, 1958. Zbl: 0080.16201.

[4] P. A. Griffiths and J. Harris. *Principles of algebraic geometry*. A Wiley-Interscience Publication. Pure and Applied Mathematics. New York, Chichester, Brisbane, Toronto, Singapore: Wiley, 1978. Zbl: 0408.14001.

[5] R. C. Gunning and H. Rossi. *Analytic functions of several complex variables*. Prentice Hall Series in Modern Analysis. Englewood Cliffs, NJ: Prentice Hall, 1965. Zbl: 0141.08601.

[6] W. V. D. Hodge and D. Pedoe. *Methods of algebraic geometry*. Vol. 2. Cambridge University Press, 1952. Zbl: 0048.14502.

[7] S.-T. Hu. *Homology theory. A first course in algebraic topology*. Holden-Day Series in Mathematics. San Francisco, London, Amsterdam: Holden-Day, 1966. Zbl: 0145.19705.

[8] K. Kodaira. 'On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties'. In: *Annals of Mathematics* 60 (1954), pp. 28–48. Zbl: 0057.14102.

[9] K. Kodaira and D. C. Spencer. 'Groups of complex line bundles over compact Kähler varieties'. In: *Proceedings of the National Academy of Sciences of the United States of America* 39 (1953), pp. 868–872. Zbl: 0051.14503.

[10] A. Newlander and L. Nirenberg. 'Complex analytic coordinates in almost complex manifolds'. In: *Annals of Mathematics* 65 (1957), pp. 391–404. Zbl: 0079.16102.

[11] N. Steenrod. *The topology of fibre bundles*. Princeton Mathematical Series 14. Princeton University Press, 1951. Zbl: 0054.07103.

[12] A. Van de Ven. 'On the Chern numbers of certain complex and almost complex manifolds'. In: *Proceedings of the National Academy of Sciences of the United States of America* 55 (1966), pp. 1624–1627. Zbl: 0144.21003.

Index